PROBLEM OF THE WEEK
Solution of Problem No. 6 (Spring 2007 Series)

**Problem:** Let \( E \) be an ellipse which is not a circle. Among all inscribed rectangles, show that

(a) exactly one is a square, and
(b) at least one has greater area than the square one.

**Solution** (by Louis J. Cote, Emeritus Professor of Statistics)

Let \( E \) in standard position have the equation, \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) with \( a \neq b \). Most older texts for courses in analytic geometry show that the locus of midpoints of a family of parallel chords with slope \( m \) is a straight line through the center of the ellipse with slope \( -b^2/(a^2m) = m' \), a diameter of the ellipse. A system of chords of slope \( m' \) similarly gives rise to a conjugate diameter of slope \( m \).

For any chord in the slope \( m \) system there is a chord of equal length symmetrically placed on the other side of the center. The end points of these two are vertices of a parallelogram inscribed in the ellipse. Every inscribed parallelogram may be gotten this way. Since the line connecting the midpoints of opposite sides of a parallelogram is parallel to the other sides, the latter have the conjugate slope, \( m' \). Since \( m \) and \( m' \) have opposite signs, conjugate diameters cannot be in the same quadrant.

For inscribed rectangles the slopes, \( m, m' \), are perpendicular, \( mm' = -1 = -b^2/a^2 \). Because \( a \neq b \), this equation must be degenerate, so one of \( m, m' \) must be zero. Thus every inscribed rectangle’s sides are parallel to the axes of \( E \) and opposite sides are on opposite sides of the axes. If the rectangle’s vertex in the first quadrant is \( (x_1, y_1) \), its area is \( 4x_1y_1 \) whose critical point under the constraint that \( (x_1, y_1) \) be on \( E \) is reached when \( x_1b = y_1a \), as may be found by calculus. The minimal area is for a rectangle with either \( y_1 \) or \( x_1 \) equal to zero at which this equation becomes degenerate, so the critical point is a maximum. The largest inscribed rectangle has its quadrant \( I \) vertex at the intersection of \( E \) with \( y = bx/a \). Since \( a \neq b \), this is not the vertex of a square which answers (b). There is only one square, that with its quadrant \( I \) vertex at the intersection of \( E \) with the line \( y = x \), which answers (a).

Also solved by:

**Undergraduates:** Noah Blach

**Graduates:** Tomek Czajka (CS)
Others: Mark Crawford (Waubonsee Community College instructor), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics)