PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2007 Series)

Problem: Let E be an ellipse which is not a circle. Among all inscribed rectangles, show that

(a) exactly one is a square, and

(b) at least one has greater area than the square one.

Solution (by Louis J. Cote, Emeritus Professor of Statistics)

Let E in standard position have the equation, $x^2/a^2 + y^2/b^2 = 1$ with $a \neq b$. Most older texts for courses in analytic geometry show that the locus of midpoints of a family of parallel chords with slope m is a straight line through the center of the ellipse with slope $-b^2/(a^2m) = m'$, a diameter of the ellipse. A system of chords of slope m' similarly gives rise to a conjugate diameter of slope m.

For any chord in the slope m system there is a chord of equal length symmetrically placed on the other side of the center. The end points of these two are vertices of a parallelogram inscribed in the ellipse. Every inscribed parallelogram may be gotten this way. Since the line connecting the midpoints of opposite sides of a parallelogram is parallel to the other sides, the latter have the conjugate slope, m'. Since m and m' have opposite signs, conjugate diameters cannot be in the same quadrant.

For inscribed rectangles the slopes, m, m', are perpendicular, $mm' = -1 = -b^2/a^2$. Because $a \neq b$, this equation must be degenerate, so one of m, m' must be zero. Thus every inscribed rectangle's sides are parallel to the axes of E and opposite sides are on opposite sides of the axes. If the rectangle's vertex in the first quadrant is (x_1, y_1), its area is $4x_1y_1$ whose critical point under the constraint that (x_1, y_1) be on E is reached when $x_1b = y_1a$, as may be found by calculus. The minimal area is for a rectangle with either y_1 or x_1 equal to zero at which this equation becomes degenerate, so the critical point is a maximum. The largest inscribed rectangle has its quadrant I vertex at the intersection of E with $y = bx/a$. Since $a \neq b$, this is not the vertex of a square which answers (b). There is only one square, that with its quadrant I vertex at the intersection of E with the line $y = x$, which answers (a).

Also solved by:

Undergraduates: Noah Blach

Graduates: Tomek Czajka (CS)

Others: Mark Crawford (Waubonsee Community College instructor), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics)