PROBLEM OF THE WEEK
Solution of Problem No. 9 (Spring 2007 Series)

Problem: Let \( f(x) = \frac{2x}{1 + e^{2x}} \). Show that the \( n \)-th in derivative \( f^{(n)}(0) \) is an integer for all \( n \geq 0 \).

Solution (by the Panel)

Given \( f(x) = \frac{2x}{1 + e^{2x}} \); let \( g(x) = x \); \( h(x) = \frac{2}{1 + e^{2x}} \). Then we have

\[
\begin{align*}
  f(x) &= g(x) \cdot h(x) \\
  f^{(1)}(x) &= g'(x) \cdot h(x) + h'(x) \cdot g(x) \\
  f^{(2)}(x) &= g''(x) \cdot h(x) + 2h'(x) \cdot g'(x) + h''(x) \cdot g(x) \\
  f^{(3)}(x) &= g'''(x) \cdot h(x) + 3g''(x) \cdot h'(x) + 3h''(x) \cdot g'(x) + h'''(x) \cdot g(x) 
\end{align*}
\]

Thus, from induction we have,

\[
f^{(n)} = \sum_{k=0}^{n} c^n_k h^{(k)}(x) \cdot g^{(n-k)}(x).
\]

Therefore \( f^{(n)}(0) \) is an integer if \( c^n_k h^{(k)}(0) g^{(n-k)}(0) \)'s are all integers. Note that \( c^n_k \) is a binomial coefficient which is an integer and \( g^{(n-k)}(x) = x \) if \( n - k = 0 \), \( g^{(n-k)}(x) = 1 \) if \( n - k = 1 \), and \( g^{(n-k)}(x) = 0 \) if \( n - k \geq 1 \), hence \( g^{(n-k)}(0) \)'s are all integers. To solve our problem it suffices to show that \( h^{(k)}(0) \) are all integers.

We claim that \( h^{(k)}(x) = (-1)^k 2^{k+1}(1 + e^{2x})^{-(k+1)} r_k(e^{2x}) \) where \( r_k(x) \) is a suitable polynomial with integer coefficients for \( k = 0, 1, 2, \ldots \).

Proof of the claim: (1) For \( k = 0 \), the claim is clear by taking \( r_0(e^{2x}) = 1 \). In general, let us assume that it is true for some \( k \). Note that \( r_k^{(1)}(e^{2x}) \) is either 0 if the polynomial is a constant, or with all coefficients even. In any case we may take \( r_k^{(1)}(e^{2x}) = 2s_k^{(1)}(e^{2x}) \) with \( s_k(x) \) a polynomial with integer coefficients. Then we have

\[
\begin{align*}
  h^{(k+1)}(x) &= (-1)^k 2^{k+1}(-(k+1))(1 + e^{2x})^{-(k+2)} e^{2x} r_k(e^{2x}) + (-1)^k 2^{k+1}(1 + e^{2x})^{-(k+1)} r_k^{(1)}(e^{2x}) \\
  &= (-1)^{k+1} 2^{k+2}(1 + e^{2x})^{-(k+2)} [((k+1)(e^{2x})r_k(e^{2x}) - (1 + e^{2x})s_k(e^{2x})] \\
  &= (-1)^{k+1} 2^{k+2}(1 + e^{2x})^{-(k+2)} r_{k+1}(e^{2x}).
\end{align*}
\]

Now our claim is proved. It is easy to see that \( h^{(k)}(0) = (-1)^{k+1} 2^{k+1} r_k(1)/2^{k+1} \) an integer.
Also solved by:

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