PROBLEM OF THE WEEK
Solution of Problem No. 12 (Spring 2008 Series)

Problem: Through each vertex of a given tetrahedron T draw a plane parallel to the opposite face. Let T' be the tetrahedron bounded by these planes. Show that the lines joining the vertices of T to the corresponding vertices of T' intersect in a single point.Minghua Lin and Jinzhong Li (Shannxi Normal Univ., China),

Solution (by Pete Kornya, Ivy Tech faculty, Bloomington, IN)

Let A, B, C, D and A', B', C', D' be the vertices of T, respectively the corresponding vertices of T'. Choose a rectangular coordinate system with D at the origin, and with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{0}$ the position vectors of A, B, C, D respectively. The plane through A parallel to the opposite face is the locus of points with position vector $\mathbf{a} + s\mathbf{b} + t\mathbf{c}$ with s, t in R. In particular, $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is on this plane. Similarly, $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is on the planes through B, respectively C and parallel to the face of T opposite B, respectively C. Therefore $\mathbf{a}+\mathbf{b}+\mathbf{c}$ is the position vector of D'. Then the line joining D to D' is the locus of points with position vector $s(\mathbf{a} + \mathbf{b} + \mathbf{c})$. In particular, the point p with position vector $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ is on the line joining D to D'.

Now shift the coordinate system by $\mathbf{a}$, so that A is now at the origin, and the position vectors of A, B, C and D are now respectively $\mathbf{0}, \mathbf{b} - \mathbf{a}, \mathbf{c} - \mathbf{a}$ and $-\mathbf{a}$. By the previous argument, the point with position vector $\frac{1}{4}\Big[\mathbf{b}-\mathbf{a}+\mathbf{c}-\mathbf{a}+(-\mathbf{a})\Big]$ in the shifted coordinate system is on the line joining A to A'. Therefore the point p, with position vector $\mathbf{a}+\frac{1}{4}\Big[\mathbf{b}-\mathbf{a}+\mathbf{c}-\mathbf{a}+(-\mathbf{a})\Big] = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ in the original coordinate system, is on the line joining A to A' as well. Similarly, p is on the line joining B to B' and C to C', as required.

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