PROBLEM OF THE WEEK

Solution of Problem No. 7 (Spring 2008 Series)

Problem: Let x, y and z be real numbers in the interval $[-2, 1]$ such that $x + y + z = 0$. Show that $x^2 + y^2 + z^2 \leq 6$.

Solution (by Kevin Ventullo, Junior, IIT, Chicago)

If $x = y = z = 0$, then we are done. If this is not the case, then there must be at least one positive term and one negative term. Suppose there are exactly two negative terms (say x and y). Since z must be nonnegative, $|z| \leq 1$.

$$x + y + z = 0$$
$$x + y = -z$$
$$|x + y| = |-z| = |z|$$
$$|x + y| \leq 1 \Rightarrow |x| \leq 1 \text{ and } |y| \leq 1, \text{ since } x \text{ and } y \text{ have the same sign.}$$

We then have $x^2 + y^2 + z^2 \leq (1)^2 + (1)^2 + (1)^2 = 3 < 6$.

Suppose that exactly one term is negative (say x). Since y and z must be nonnegative, $|y| \leq 1$, $|z| \leq 1$.

Thus, $x^2 + y^2 + z^2 \leq (2)^2 + (1)^2 + (1)^2 = 6$.

This problem was suggested by Peter Montgomery of Microsoft whose solution is as follows:

$$x^2 + y^2 + z^2 = 6 - (2 + x)(1 - x) - (2 + y)(1 - y) - (2 + z)(1 - z) - (x + y + z) \leq 6.$$

Also solved by:

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Problem 6 was also solved by Al–Sharif Talal Al–Housseiny (Shell Chemical, Norco, LA).