PROBLEM OF THE WEEK
Solution of Problem No. 12 (Spring 2009 Series)

Problem: For how many positive integers $x \leq 10,000$ is $2^x - x^2$ not divisible by 7? Justify your answer without the use of computers.

Solution (by David Elden, Sophomore, Mechanical Engineering, Purdue University)

Observe that $2n(\text{mod } 7) = 2$ for $n(\text{mod } 7) = 1, 4$ for $n(\text{mod } 7) = 2$, and 1 for $n(\text{mod } 7) = 4$. So $2^n(\text{mod } 7)$ is periodic with a period of 3. Also, note that $n^2(\text{mod } 7) = (n+7)^2(\text{mod } 7)$ because $(n+7)^2 = n^2 + 14n + 49$. So $n^2(\text{mod } 7)$ is periodic with a period of 7. Thus, $2^n - n^2(\text{mod } 7)$ is periodic with a period of 21. Now, $2^x - x^2$ is divisible by 7 when $2^n - n^2(\text{mod } 7) = 0$, and is not divisible by 7 otherwise. It is trivial to confirm that of the first 21 values for $2^n - n^2(\text{mod } 7)$, 6 are zero, and 15 are not. Now, $10000 = 476 \times 21 + 4$, and the first four values of $2^n - n^2(\text{mod } 7)$ contain 2 zeros, so there are $476 \times 15 + 2 = 7142$ values in the first 10000 that are not divisible by 7.

The problem was also solved by:

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