PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2009 Series)

**Problem:** Determine positive integers $a, b, c$ so that the equation $ax^2 - bx + c = 0$ has 2 distinct real roots in the interval $0 < x < 1$ and $(a + b + c)$ is smallest possible. Your answer must be justified without the use of computers.

**Solution** (by Phuong Thanh Tran, Graduate student, ECE, Purdue University)

Let $a, b, c$ be the positive integers satisfying the given requirements. Then we have:

The equation $ax^2 - bx + c = 0$ (**) has 2 distinct roots $\iff b^2 - 4ac > 0 \iff b > 2\sqrt{ac}$ (1)

Let $x_1 < x_2$ be 2 distinct roots of (**). Then:

$$x_1 = \frac{b - \sqrt{b^2 - 4ac}}{2a} < \frac{b + \sqrt{b^2 + 4ac}}{2a} = x_2.$$

So $x_2 < 1 \iff b + \sqrt{b^2 - 4ac} < 2a \iff \sqrt{b^2 - 4ac} < 2a - b \iff \begin{cases} 2a - b > 0 \\ b^2 - 4ac < (2a - b)^2 \end{cases}$

$$\iff \begin{cases} b < 2a \\ 4a^2 - 4ab + 4ac > 0 \end{cases} \iff \begin{cases} b < 2a \\ b < a + c \end{cases}$$ (2)

From (1), (2), we get $2a > 2\sqrt{ac} \Rightarrow a > c \Rightarrow a = c + k$ where $k$ is a positive integer. (3)

Now $a + c > b > 2\sqrt{ac}$ (from (1), (2)) $\Rightarrow 2c + k > b > 2\sqrt{c(c+k)} \Rightarrow 2c + k - 1 > b > 2\sqrt{c(c+k)} \Rightarrow (2c + k - 1)^2 > 4c(c+k) \Rightarrow 4c < (k-1)^2 \Rightarrow k > 1 + 2\sqrt{c} \Rightarrow k \geq 2 + \lfloor 2\sqrt{c} \rfloor$ (4)

$$b > 2\sqrt{c(c+k)} \Rightarrow b \geq 1 + \lfloor 2\sqrt{c(c+k)} \rfloor \geq 1 + \lfloor 2\sqrt{c(c+2 + \lfloor 2\sqrt{c} \rfloor)} \rfloor$$ (5)

From (3), (4), (5), we have $a + b + c = 2c + k + b \geq 2c + 2 + \lfloor 2\sqrt{c} \rfloor + 1 + \lfloor 2\sqrt{c(c + 2 + \lfloor 2\sqrt{c} \rfloor)} \rfloor \Rightarrow a + b + c \geq 2c + 2 + \lfloor 2\sqrt{c} \rfloor + \lfloor 2\sqrt{c(c + 2 + \lfloor 2\sqrt{c} \rfloor)} \rfloor$ (6)

The right hand side of (6) is minimized when $c$ is minimized, i.e., $c = 1$. So

$$a + b + c \geq 2 + 3 + \lfloor 2 \rfloor + \lfloor 2\sqrt{1(1+2 + \lfloor 2 \rfloor)} \rfloor = 7 + \lfloor 2\sqrt{5} \rfloor = 11$$

The equality occurs when $k = 2 + \lfloor 2\sqrt{c} \rfloor$, $a = c + k$ and $b = 1 + \lfloor 2\sqrt{c(c + k)} \rfloor \iff k = 4$, $a = 5, b = 5$. We can verify that (**) has 2 distinct roots:

$$x_1 = \frac{5 - \sqrt{5}}{10}, x_2 = \frac{5 + \sqrt{5}}{10} \text{ and } 0 < \frac{5 - \sqrt{5}}{10} < \frac{5 + \sqrt{5}}{10} < \frac{5 + 5}{10} = 1.$$
So \( a + b + c \geq 11 \) and the equality occurs \( \iff a = b = 5 \) and \( c = 1 \).

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