PROBLEM OF THE WEEK

Solution of Problem No. 3 (Spring 2009 Series)

Problem: Determine positive integers a, b, c so that the equation $ax^2 - bx + c = 0$ has 2 distinct real roots in the interval $0 < x < 1$ and $(a + b + c)$ is smallest possible. Your answer must be justified without the use of computers.

Solution (by Phuong Thanh Tran, Graduate student, ECE, Purdue University)

Let a, b, c be the positive integers satisfying the given requirements. Then we have:

The equation $ax^2 - bx + c = 0 \quad (*)$ has 2 distinct roots $\Leftrightarrow b^2 - 4ac > 0 \Leftrightarrow b > 2\sqrt{ac} \quad (1)$

Let $x_1 < x_2$ be 2 distinct roots of (*). Then:

$$x_1 = \frac{b - \sqrt{b^2 - 4ac}}{2a} < \frac{b + \sqrt{b^2 + 4ac}}{2a} = x_2.$$

So $x_2 < 1 \Leftrightarrow b + \sqrt{b^2 - 4ac} < 2a \Leftrightarrow \sqrt{b^2 - 4ac} < 2a - b \Leftrightarrow \begin{cases} 2a - b > 0 \\ b^2 - 4ac < (2a - b)^2 \end{cases}$

$$\Leftrightarrow \begin{cases} b < 2a \\ 4a^2 - 4ab + 4ac > 0 \end{cases} \Leftrightarrow \begin{cases} b < 2a \\ b < a + c \end{cases} \quad (2)$$

From (1), (2), we get $2a > 2\sqrt{ac} \Rightarrow a > c \Rightarrow a = c + k$ where k is a positive integer. (3)

Now $a + c > b > 2\sqrt{ac}$ (from (1), (2)) $\Rightarrow 2c + k > b > 2\sqrt{c(c + k)} \Rightarrow 2c + k - 1 \geq b > 2\sqrt{c(c + k)} \Rightarrow (2c + k - 1)^2 > 4c(c + k) \Rightarrow 4c < (k - 1)^2 \Rightarrow k > 1 + 2\sqrt{c} \Rightarrow k \geq 2 + \lfloor 2\sqrt{c} \rfloor \quad (4)$

$b > 2\sqrt{c(c + k)} \Rightarrow b \geq 1 + \lfloor 2\sqrt{c(c + k)} \rfloor \geq 1 + \lfloor 2\sqrt{c(c + 2 + \lfloor 2\sqrt{c} \rfloor)} \rfloor \quad (5)$

From (3), (4), (5), we have $a+b+c = 2c+k+b \geq 2c+2+\lfloor 2\sqrt{c} \rfloor+1+\lfloor 2\sqrt{c(c + 2 + \lfloor 2\sqrt{c} \rfloor)} \rfloor \Rightarrow a + b + c \geq 2c + 3 + \lfloor 2\sqrt{c} \rfloor + \lfloor 2\sqrt{c(c + 2 + \lfloor 2\sqrt{c} \rfloor)} \rfloor \quad (6)$

The right hand side of (6) is minimized when c is minimized, i.e., $c = 1$. So

$$a + b + c \geq 2 + 3 + \lfloor 2 \rfloor + \lfloor 2\sqrt{1(1 + 2 + \lfloor 2 \rfloor)} \rfloor = 7 + \lfloor 2\sqrt{5} \rfloor = 11$$

The equality occurs when $k = 2 + \lfloor 2\sqrt{c} \rfloor$, $a = c + k$ and $b = 1 + \lfloor 2\sqrt{c(c + k)} \rfloor \Leftrightarrow k = 4$, $a = 5, b = 5$. We can verify that (*) has 2 distinct roots:

$$x_1 = \frac{5 - \sqrt{5}}{10}, x_2 = \frac{5 + \sqrt{5}}{10} \text{ and } 0 < \frac{5 - \sqrt{5}}{10} < \frac{5 + \sqrt{5}}{10} < \frac{5 + 5}{10} = 1.$$

So $a + b + c \geq 11$ and the equality occurs $\Leftrightarrow$ $a = b = 5$ and $c = 1$.

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