**Problem of the Week**
Solution of Problem No. 9 (Spring 2009 Series)

**Problem:** If \( n \) is a given positive integer, how many solutions \((x, y)\) does

\[
\frac{1}{n} = \frac{1}{x} + \frac{1}{y}
\]

have with \( x \) and \( y \) unequal positive integers?

**Solution** (by Gruian Cornel, IT, Romania)

If \((x, y)\) is a solution for \(\frac{1}{x} + \frac{1}{y} = \frac{1}{n}\), clearly \(\min(x, y) > n\). If not, say \(\min(x, y) = x \leq n\) then \(\frac{1}{n} = \frac{1}{x} + \frac{1}{y} > \frac{1}{n}\), contradiction. So \(x > n\) and \(y > n\). We write the equation as \(n(x+y) = xy\) or \((x-n)(y-n) = n^2\) and for \(r\) positive \(r|n^2\), the solutions are given by \(x-n = r\) and \(y-n = \frac{n^2}{r}\) or \(x = n + r\) and \(y = n + \frac{n^2}{r}\). The only case when \(x = y\) is when \(r = \frac{n^2}{r}\) or \(r = n\) and the numbers of solutions \((x, y)\) with \(x \neq y\) is \(d(n^2) - 1\) where \(d(n^2)\) is the number of divisors of \(n^2\), \(d(n^2) = (2q_1 + 1)(2q_2 + 1) \ldots (2q_n + 1)\), where \(n = p_1^{q_1} p_2^{q_2} \ldots p_m^{q_m}\) is the prime factorization of \(n\).

The problem was also solved by:

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