PROBLEM OF THE WEEK

Solution of Problem No. 1 (Spring 2010 Series)

Problem: If the integers from 1 to 222,222,222 are written down in succession, how many of them have at least one zero?

Solution (by Yixin Wang, Freshman, Purdue University)

Let's count how many numbers there are from 1 to 222,222,222 that don't have any $0's$. We'll categorize them by how many digits they have:

For n–digit numbers, 9^n of them don't have any 0's. (Each of the n digits can be a number from 1 through 9, so that's 9 choices for each of the n digits.) So, for 1– through 8–digit numbers, there are a total of $9^1 + 9^2 + \cdots + 9^8 = \dfrac{9^9 - 1}{9 - 1} - 1$ numbers with no 0's.

For 9–digit numbers, we only need to consider the numbers 111,111,111 to 222,222,222. For these, we further categorize them by how many 2's they have in a row starting from the left (For example, 222217687 starts with four 2's in a row, and 219825675 starts with one 2). Consider all 9–digit numbers that start with exactly k 2's, with $0 \le k \le 8$. That means its next digit can't be 2 (since that creates a number with $k+1$ 2's in a row), so the next digit has to be 1. Then, the other $9-k-1$ digits after that can be any of the numbers 1 through 9, so that gives 9^{9-k-1} choices. This means that there are 9^{9-k-1} 9–digit numbers that start with k 2's and don't have any 0's. Since k takes on the values 0 through 8, the total number of these is $9^8 + 9^7 + \cdots + 9^1 + 9^0$. However, because $0 \le k \le 8$, we haven't counted numbers that start with 9 2's. There's only 1 of those, so we simply add 1 to our total, so there are actually $9^8 + 9^7 + \cdots + 9^1 + 9^0 + 1 = \dfrac{9^9 - 1}{9 - 1} + 1$ nine–digit numbers from 111,111,111 to 222,222,222 with no 0's.

So, our grand total comes to $\dfrac{9^9 - 1}{9 - 1} - 1 + \dfrac{9^9 - 1}{9 - 1} + 1 = \dfrac{9^9 - 1}{4}$. But that's the number of integers WITHOUT 0's, so we need to subtract that from 222,222,222:

$$222222222 - \frac{9^9 - 1}{4} = 125367100$$

The problem was also solved by:

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