**Problem of the Week**

Solution of Problem No. 13 (Spring 2010 Series)

**Problem:** Show that for $0 < \varepsilon < 1$ the expression $(x+1)^n(x^2-(2-\varepsilon)x+1)$ is a polynomial with strictly positive coefficients if $n$ is sufficiently large. For $\varepsilon = 10^{-3}$ find the smallest possible $n$.

**Solution** (by Gruian Cornel, IT, Romania)

Let $p(x) = (x+1)^n(x^2-(2-\varepsilon)x+1) = \sum_{k=0}^{n+2} a_k x^{n+2-k}$ where $a_0 = a_{n+2} = 1$, $a_1 = a_{n+1} = n-(2-\varepsilon)$ and $a_{k+2} = \binom{n}{k+2} - (2-\varepsilon) \binom{n}{k+1} + \binom{n}{k}$ for $k = 0, 1, \ldots, n-2$. For $a_j > 0$, we have the conditions $\frac{n-k}{(k+1)(k+2)} - (2-\varepsilon) \frac{n-k}{k+1} + 1 > 0$, or $(n+1) \left( \frac{1}{k+2} + \frac{1}{n-k} \right) > 4 - \varepsilon$ where $k = 0, 1, \ldots, n-2$. We need that (1): $(n+1) \min \left\{ \left( \frac{1}{k+2} + \frac{1}{n-k} \right) : k = 0, 1, \ldots, n-2 \right\} > 4 - \varepsilon$. Consider $f : [0, n-2] \to (0, \infty)$, $f(x) = \frac{1}{x+2} + \frac{1}{n-x}$, $f'(x) = \frac{(n+2)(2x-(n-2))}{(x+2)^2(n-x)^2}$, $f'(x) < 0$ on $\left[ 0, \frac{n-2}{2} \right]$, $f'(x) > 0$ on $\left( \frac{n-2}{2}, n-2 \right]$, $\frac{n-2}{2}$ is a minimum point for $f$ and $f\left( \frac{n-2}{2} \right) = \frac{4}{n+2}$. Hence if $\frac{4(n+1)}{n+2} > 4 - \varepsilon$, or $n > \frac{4}{\varepsilon} - 2$ then $a_j > 0$ for $j = 0, 1, \ldots, n+2$. Now we inspect the cases:

1) For $n$ even, $n = 2m$ then $\min \left\{ \left[ \frac{1}{k+2} + \frac{1}{n-k} \right] : k = 0, \ldots, n-2 \right\} = f(m-1) = \frac{2}{m+1}$ and (1) becomes $\frac{2(2m+1)}{m+1} > 4 - \varepsilon$, or $m > \frac{2}{\varepsilon} - 1$. For $\varepsilon = 10^{-3}$, $m_{\text{min}} = 2 \cdot 10^3$ and $n_{\text{min}} = 4000$.

2) For $n$ odd, $n = 2m+1$ then $\min \left\{ \left[ \frac{1}{k+2} + \frac{1}{n-k} \right] : k = 0, \ldots, n-2 \right\} = f(m) = f(m-1) = \frac{1}{m+1} + \frac{1}{m+2}$ and (1) becomes $\frac{2(2m+3)}{m+2} > 4 - \varepsilon$, or $m > \frac{2}{\varepsilon} - 2$. For $\varepsilon = 10^{-3}$, $m_{\text{min}} = 2 \cdot 10^3 - 1$ and $n_{\text{min}} = 4000 - 1 = 3999$.

Hence for $\varepsilon = 10^{-3}$ the smallest possible $n$ is 3999.

Also completely or partially solved by:
**Undergraduates:** Kilian Cooley (Fr.)

**Graduates:** Richard Eden (Math), Tairan Yuwen (Chemistry)

**Others:** Neacsu Adrian (Romania), Siavash Ameli (Grad. student, Toosi Univ. of Tech, Iran), Hongwei Chen (Christopher Newport U. VA), Sandipan Dey (Graduate student, UMBC), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Sorin Rubinstein (TAU faculty, Israel) Craig Schroeder (Grad student, Stanford Univ.), Steve Spindler (Chicago), Zhengpeng Wu (Engr. Tsinghua Univ. China)