Problem: A sequence \(a_0, a_1, a_2, \ldots\) of real numbers satisfies

(1) \(0 \leq a_0 \leq 1\)

and

(2) \(a_{n+1} = 4a_n^3 - 6a_n^2 + a_n + 1\) \((n = 0, 1, 2, \ldots)\).

Given that \(\lim_{n \to \infty} a_n\) exists, find (with proof) the possible value(s) of \(a_0\).

Solution  
(by Craig Schroeder, Ph.D. student, Stanford University)

Let \(f(x) = 4x^3 - 6x^2 + x + 1\). Let \(a\) be such a limit. Then, \(a = f(a)\). This has three solutions: \(a = \frac{1}{2}, a = \frac{1}{2} \pm \frac{1}{2}\sqrt{3}\).

Let \(r = \frac{1}{2} - \frac{2}{9}\sqrt{6}\) and \(s = \frac{1}{2} + \frac{2}{9}\sqrt{6}\). Consider the interval \(I = [r, s]\). Iteration starts in this interval, since \([0, 1] \subset I\). The extreme values of \(f\) occur at the endpoints or at local extrema. \(f(r) = \frac{1}{2} + \frac{44}{243}\sqrt{6} \in I\) and \(f(s) = \frac{1}{2} - \frac{44}{243}\sqrt{6} \in I\). \(f'(x) = 12x^2 - 12x + 1\), so the critical points are \(c \pm = \frac{1}{2} \pm \frac{1}{6}\sqrt{6}\), so that \(f(c_-) = s\) and \(f(c_+) = r\). Thus, \(f(I) = I\). Since \(\frac{1}{2} \pm \frac{1}{2}\sqrt{3} \notin I\), no valid starting point can converge to those values. Thus, any sequence that converges must converge to \(\frac{1}{2}\).

The initial value \(a_0 = \frac{1}{2}\) leads trivially to a constant sequence that converges. The other two solutions to \(f(x) = \frac{1}{2}\) lie outside \(I\). The other possibility is that the sequence converges to \(\frac{1}{2}\) without actually obtaining that value. Let \(a_n = \frac{1}{2} + \epsilon\), so that \(a_{n+1} = \frac{1}{2} - 2\epsilon + 4\epsilon^3\). Assume that \(|\epsilon| < \frac{1}{4}\), so that \(\frac{1}{2} - a_{n+1} = 2|\epsilon||1 - 2\epsilon^2| > \frac{7}{4}|\epsilon| > |\epsilon|\). Since the sequence diverges from \(\frac{1}{2}\), there are no other converging sequences. The only possible starting value is \(a_0 = \frac{1}{2}\).

Also completely or partially solved by:
Undergraduates: Yixin Wang (Fr.)

Graduates: Tairan Yuwen (Chemistry)

Others: Neacsu Adrian (Romania), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Tom Engelsman (Chicago, IL), Nathan Faber (CO), Elie Ghosn (Montreal, Quebec), Steven Landy (IUPUI Physics staff), Sorin Rubinstein (TAU faculty, Israel) Steve Spindler (Chicago), Thierry Zell (Ph.D, Purdue 03)