PROBLEM OF THE WEEK
Solution of Problem No. 4 (Spring 2011 Series)

**Problem:** Show that four consecutive binomial coefficients

\[
\binom{n}{r}, \binom{n}{r+1}, \binom{n}{r+2}, \binom{n}{r+3}
\]

(with \(n, r\) positive and \(r + 3 \leq n\)) can never be in arithmetic progression.

**Solution:** (by Steven Landy, IUPUI Physics staff)

If we try to find three terms in arithmetic progression we need,

\[
\binom{n}{r} - \binom{n}{r+1} = \binom{n}{r+1} - \binom{n}{r+2}.
\]  

(1)

When (1) is expanded into factorials and reduced we find

\[
(r + 1)(r + 2) - 2(n - r)(r + 2) + (n - r)(n - r - 1) = 0.
\]

(2)

Equation (2) may be solved as a quadratic for either \(n\) or \(r\) giving

\[
n = \frac{1}{2}(4r + 5 \pm \sqrt{8r + 17}) \quad \text{and} \quad r = \frac{1}{2}(n - 2 \pm \sqrt{n + 2}).
\]

So there are many solutions for three consecutive binomial coefficients in arithmetic progression, for example \(n = 7, r = 1\) or 4. These are actually the same three numbers but in reverse order. A “four in a row” would require a second “three in a row” beginning one space to the right of another three in a row. Since for any legal \(n\) there are only two acceptable \(r\) values (which are not consecutive) there can never be four consecutive binomial coefficients in arithmetic progression.

The problem was also solved by:

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