PROBLEM OF THE WEEK

Solution of Problem No. 6 (Spring 2011 Series)

Problem: The possible scores in the game of tossing two dice are the integers $2, 3, \ldots, 12$.

Is it possible to load the dice in such a way that these eleven scores are equally probable?

Remark. "Loading" the dice means assigning probabilities to each of the six sides coming up. The two dice do not have to be loaded in the same way.

Solution 1: (by Landon Lehman, Senior, Physics, Purdue University)

Let the two dice be called A and B. Let a be the probability that die A rolls a 1, and let b be the probability that die B rolls a 1. Similarly, let c be the probability that die A rolls a 6, and let d be the probability that die B rolls a six.

Then the probability of getting a score of 12 when tossing the two dice is cd, and the probability of getting a score of 2 is ab. If the dice are loaded in such a way as to make all 11 scores equally probable, $ab = 1/11$ and $cd = 1/11$. The probability of getting a score of 7 is $ad + bc + \text{otherterms} = 1/11$. So

$$ad + bc \leq \frac{1}{11}.$$

We can rewrite this as

$$\frac{a}{11c} + \frac{c}{11a} \leq \frac{1}{11}.$$

But it is easy to show that, for any two positive numbers x and y

$$\frac{z}{y} + \frac{y}{x} \geq 2.$$

This means that

$$\frac{a}{11c} + \frac{c}{11a} \geq \frac{2}{11}$$

which is a contradiction. Therefore it is not possible to load the two dice so that each of the eleven scores is equally probable.

Solution 2: (by Gruian Cornel, Cluj–Napoca, Romania)

Let us assign probabilities x_i to the side i of the first die and y_i to the side j of the second one. Now the probability to obtain the score k is $p_k = \sum_{i+j=k} x_i y_j$, $k \in \{2, 3, \ldots, 12\}$.

Suppose that $p_2 = p_3 = \cdots = p_{12} = p \in (0,1]$. Consider the polynomials $P, Q, R \in \mathbb{C}[Z]$ where $P(z) = x_1 + x_2 z + \cdots + x_6 z^5$, $Q(z) = y_1 + y_2 z + \cdots + y_6 z^5$ and $R(z) = 1 + z + z^2 + \cdots + z^{10}$. Therefore for any $z \in \mathbb{C}$, $P(z)Q(z) = \sum_{r=0}^{10} \left(\sum_{i+j=r+2} x_i y_j \right) z^r = \sum_{r=0}^{10} p_{r+2} z^r = pR(z)$. This is a contradiction because $\deg(P) = \deg(Q) = 5$ and so each of them has a real root. On the other hand the roots of R are $\varepsilon, \varepsilon^2, \ldots, \varepsilon^{10} \in \mathbb{C} \setminus \mathbb{R}$ where $\varepsilon = \cos \frac{2\pi}{11} + i \sin \frac{2\pi}{11}$. Hence it is not possible to load the dice such that the eleven scores are equally nonzero probable.

The problem was also solved by:

Undergraduates: Kaibo Gong (Math), Yixin Wang (So. ECE)

Graduates: Pinaki Bhattacharya (Mech.E), Shuhao Cao (Math), Richard Eden (Math), Benjamin Philabaum (Phys.), Jeremy Troisi (Stat), Tairan Yuwen (Chemistry)

Others: Manuel Barbero (New York), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Jeff Krimmel (Houston, TX), Steven Landy (IUPUI Physics staff), Kevin Laster (Indianapolis, IN), Paul Liu & Ron Estrin(Canada), Denes Molnar (Physics, Assistant Professor), Louis Rogliano (Corsica), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Ph.D. student, Stanford Univ.) Steve Spindler (Chicago), Stephen Taylor (Bloomberg L.P. NY), Martin Vlietstra (Software engineer, United Kingdom), William Wu (JPL)