PROBLEM OF THE WEEK
Solution of Problem No. 7 (Spring 2011 Series)

Problem: Show that
\[
\frac{1}{2\sqrt{n}} < \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} < \frac{1}{\sqrt{2n+1}}
\]
for every \( n = 2, 3, \ldots \).

Solution: (by Richard Eden, Math Graduate student, Purdue University)

For any \( k > \frac{1}{2} \),
\[
\frac{2k-1}{2k} < \frac{\sqrt{2k-1}}{\sqrt{2k+1}} \iff \sqrt{2k-1}\sqrt{2k+1} < 2k \iff 4k^2 - 1 < 4k^2,
\]
and the last inequality is true. As \( k \) runs through the integers from 1 to \( n \),
\[
\frac{1 \cdot 3 \cdot 5 \cdots 2n-1}{2 \cdot 4 \cdot 6 \cdots 2n} < \frac{\sqrt{1}}{\sqrt{3}} \frac{\sqrt{3}}{\sqrt{5}} \frac{\sqrt{5}}{\sqrt{7}} \cdots \frac{\sqrt{2n-1}}{\sqrt{2n+1}} = \frac{1}{\sqrt{2n+1}}.
\]
For any \( k > 1 \),
\[
\frac{\sqrt{k-1}}{\sqrt{k}} < \frac{2k-1}{2k} \iff 2\sqrt{k}\sqrt{k-1} < 2k-1 \iff 4k^2 - 4k < 4k^2 - 4k + 1,
\]
where the last inequality is again true. As \( k \) runs through the integers from 2 to \( n \),
\[
\frac{1 \cdot 3 \cdot 5 \cdots 2n-1}{2 \cdot 4 \cdot 6 \cdots 2n} > \frac{1}{2} \cdot \frac{\sqrt{1}}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{3}} \cdots \frac{\sqrt{n-1}}{\sqrt{n}} = \frac{1}{2\sqrt{n}}.
\]

The problem was also solved by:

Undergraduates: Kilian Cooley (So.), Sean Fancher (Science), Kaibo Gong (Math), Hai Huang (Fr. Math), Landon Lehman (Sr. Phys.), Hongshau Li (Sr. Math), Jason Macnak (So. Math), Yixin Wang (So. ECE) Lifan Wu (So.), Lirong Yuan (Fr.)

Graduates: Shuhao Cao (Math), Ankit Jain (ECE), Murali Medisetty (CS) & Siddhartha Jetti (M.E.T), Benjamin Philabaum (Phys.), Krishnaraj Sambath (Ch.E.), Tairan Yuwen (Chemistry)
Others: Manuel Barbero (New York), Hongwei Chen (Christopher Newport U. VA), Gruian Cornel (IT, Romania), Sandipan Dey (Graduate student, UMBC), Jonathan Dorfman (Bloomberg, LP, NY), Tom Engelsman (Chicago, IL), Elie Ghosn (Montreal, Quebec), Jeff Krimmel (Houston, TX), Steven Landy (IUPUI Physics staff), Denes Molnar (Physics, Assistant Professor), Angel Plaza (ULPGC, Spain), Louis Rogliano (Corsica), Craig Schroeder (Ph.D. student, Stanford Univ.) Pawan Singh (Indianapolis), Steve Spindler (Chicago), Stephen Taylor (Bloomberg L.P. NY), Daniel Tsai (Taipei American School, Taiwan), William Wu (JPL), Turkay Yolcu (Visiting at Purdue U.), Allen Zhang (Undergraduate, U. of British Columbia)