PROBLEM OF THE WEEK
Solution of Problem No. 8 (Spring 2011 Series)

Problem: Find the smallest volume bounded by the coordinate planes and by a tangent plane to the ellipsoid.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Solution: (by Elie Ghosn, Montreal, Quebec)

The coordinate planes are planes of symmetry for the ellipsoid. Therefore we can consider only the region with positive coordinates. It's easy to show that the equation of the tangent plane to the ellipsoid at $P(x_0,y_0,z_0)$ is $\frac{Xx_0}{a^2}+\frac{Yy_0}{b^2}+\frac{Zz_0}{c^2}=1$ with $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}=1$.

The region is a tetrahedron whose vertices are $(0,0,0)$, $\left(\frac{a^2}{x_0},0,0\right)$, $\left(0,\frac{b^2}{y_0},0\right)$ and $\left(0,0,\frac{c^2}{z_0}\right)$ and it's volume is $V_{P_0}=\frac{1}{6}\,\frac{a^2b^2c^2}{x_0y_0z_0}$.
From

$$\frac{x_0^2}{a^2}\cdot\frac{y_0^2}{b^2}\cdot\frac{z_0^2}{c^2}\le\left(\frac{\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}}{3}\right)^3=\frac{1}{27}\qquad\text{(Arithmetic–geometric mean inequality)}$$

with equality iff $\frac{x_0}{a}=\frac{y_0}{b}=\frac{z_0}{c}=\frac{1}{\sqrt{3}}$ we deduce $\frac{\sqrt{3}}{2}abc\le V_{P_0}$.

Therefore, the minimum volume occures at $P_0\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$ and is equal to $\frac{\sqrt{3}}{2}abc$.

The problem was also solved by:

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