Problem: A, B, C, D are four distinct points in three space. Suppose each of the angles $\angle ABC$, $\angle BCD$, $\angle CDA$, and $\angle DAB$ are right angles. Show that all four points lie in the same plane.

**Solution 1:** (by Kilian Cooley, Junior, Math & AAE, Purdue University)

Assume without loss of generality that points $A$, $B$, and $C$ are located along the $y$-axis, at the origin, and along the $x$-axis respectively of some coordinate system. Denote by $\vec{r}_{PQ}$ the position vector from point $P$ to point $Q$, which are expressed as column vectors. Thus, using the fact that angles $\angle DAB$ and $\angle BCD$ are right angles:

$$\vec{r}_{BD} = \vec{r}_{BA} + \vec{r}_{AD} = \vec{r}_{BC} + \vec{r}_{CD}$$

$$\vec{r}_{BA} \cdot \vec{r}_{AD} = 0$$

$$\vec{r}_{BC} \cdot \vec{r}_{CD} = 0$$

Since point $A$ is located on the $y$-axis and $C$ on the $x$-axis, the second and third relations can be written respectively as

$$\begin{bmatrix} 0 \\ y_A \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_D - x_A \\ y_D - y_A \\ z_D - z_A \end{bmatrix} = 0$$

$$\begin{bmatrix} x_C \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x_D - x_C \\ y_D - y_C \\ z_D - z_C \end{bmatrix} = 0$$

From which we obtain

$$y_D - y_A = 0$$

$$x_D - x_C = 0$$

So point $D$ lies along a line parallel to the $z$-axis through $\begin{bmatrix} x_C \\ y_A \\ 0 \end{bmatrix}$. We want to show now that $z_D = 0$. Since $\angle CDA$ is a right angle, it follows that

$$\vec{r}_{AD} \cdot \vec{r}_{CD} = 0$$
\[
\begin{bmatrix}
  x_D - x_A \\
  y_D - y_A \\
  z_D - z_A
\end{bmatrix}
\cdot
\begin{bmatrix}
  x_D - x_C \\
  y_D - y_C \\
  z_D - z_C
\end{bmatrix}
= 
\begin{bmatrix}
  x_D - x_A \\
  y_D - y_C \\
  z_D - z_A
\end{bmatrix}
\cdot
\begin{bmatrix}
  0 \\
  y_D - y_C \\
  z_D - z_C
\end{bmatrix}
= (z_D - z_A)(z_D - z_C) = 0
\]

Since points \( A \) and \( C \) lie in the same plane by definition, \( z_A = z_C = 0 \). Hence \( z_D = 0 \), and all four points lie in the same plane.

**Solution 2:**  (by Steven Landy, Physics Faculty, IUPUI)

From perpendicularity we have, using vectors,

\[
(B - A) \cdot (C - B) = 0 \quad \text{and} \quad (C - B) \cdot (D - C) = 0,
\]

so that

\((C - B)\) is a multiple of \((B - A) \times (D - C)\) unless \((B - A) \times (D - C) = \vec{0}\). In the same way we see that

\((A - D)\) is a multiple of \((B - A) \times (D - C)\) unless \((B - A) \times (D - C) = \vec{0}\).

Therefore either \((C - B)\) is parallel to \((A - D)\), so these vectors are coplanar or \((B - A) \times (D - C) = \vec{0}\) so \((B - A)\) is parallel to \((D - C)\), which proves the theorem.

The problem was also solved by:

**Undergraduates:** Seongjun Choi (Sr. Math), Ke Ding Fye, Kaibo Gong (Sr. Math), Hai Huang (Jr. Eco & Math)

**Graduates:** Richard Eden (Math), Paul Farias (IE), Dat Tran (Math), Tairan Yuwen (Chemistry)

**Others:** Manuel Barbero (New York), Charles Burnette (Philadelphia), Ioan Viorel Codreanu (Secondary school, Romania), Gruian Cornel (Cluj-Napoca, Romania), Hubert Desprez (Paris, France), Tom Engelsman (Tampa, FL), Talal Al Fares (Hasbaya, Nabatieh, Lebanon), Elie Ghosn (Montreal, Quebec), Sreikanth Gopalan (Professor, Boston Univ.), John Karpis (Miami Springs, FL), Wei-hsiang Lien (Research assistant, National Chiao-Tung Univ., Taiwan), Jean Pierre Mutanguha (Student, Oklahoma Christian University), Jason Rahman (High School Senior, Hazleton, IN), Sorin Rubinstein (TAU faculty, Israel), Craig Schroeder (Postdoc. UCLA), Jason L. Smith (Professor, Phys. & Math. Richland Community College)