PROBLEM OF THE WEEK
Solution of Problem No. 11 (Spring 2013 Series)

Problem:
Let \( c_0 > 0 \), \( c_1 > 0 \), and \( c_{n+1} = \sqrt{c_n} + \sqrt{c_{n-1}}, \ n \geq 1 \).
Show that \( \lim_{n \to \infty} c_n \) exists and find this limit.

Solution: (by Julien Bureaux, Paris, France)

Let \( c_0 > 0 \), \( c_1 > 0 \), and
\[
    c_{n+1} = \sqrt{c_n} + \sqrt{c_{n-1}}, \quad n \geq 1 \tag{1}
\]
Show that \( \lim_{n \to \infty} c_n \) exists and find this limit.

We will prove that
\[
    \lim \sup c_n \leq 4 \leq \lim \inf c_n \tag{2}
\]
First remark that the sequence \( b_n = \max\{4, c_n, c_{n-1}\} \) is non-increasing. Indeed, the trivial lower bound \( b_n \geq 4 \) yields \( c_{n+1} \leq 2\sqrt{b_n} \leq b_n \); we conclude with \( b_{n+1} = \max\{4, c_{n+1}, c_n\} \leq \max\{4, b_n, b_n\} = b_n \). As a consequence, an upper bound for \( c_n \) is \( \max\{4, c_0, c_1\} \). In the same way, \( c_n \geq \min\{4, c_0, c_1\} \).

These bounds show that both \( \lim \inf c_n \) and \( \lim \sup c_n \) lie in \((0, \infty)\). Furthermore we deduce from (1) that
\[
    \lim \inf c_n \geq 2 \sqrt{\lim \inf c_n}, \quad \lim \sup c_n \leq 2 \sqrt{\lim \sup c_n}
\]
This proves (2), hence the result.

The problem was also solved by:

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