Problem:

Find the maximum possible value of $\prod_{i=1}^{n} a_i$ given $0 \leq a_i \leq i$, $\sum_{i=1}^{n} a_i = 3n$.

Solution: (by Sorin Rubinstein, TAU faculty, Tel Aviv, Israel)

Let $H_n = ([0,1] \times [0,2] \times \cdots \times [0,n]) \cap \{x_1, x_2, \ldots, x_n : \sum_{i=1}^{n} x_i = 3n\}$ and $\varphi : H_n \to R$, $\varphi(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i$. Since $1 + 2 + \cdots + n = \frac{n(n+1)}{2} \geq 3n$ iff $n \geq 5$ it follows that $H_n$ is void for $n = 1, 2, 3, 4$ and consists of the single point: $(1, 2, 3, 4, 5)$ if $n = 5$. Assume that $n \geq 6$. Since $H_n$ is compact, the continuous function $\varphi$ attains its maximal value on some point of $H_n$. Let $a = (a_1, a_2, \ldots, a_n)$ be such a point. Clearly, $\varphi$ is not identical 0.

Hence $a_i > 0$, $i = 1, 2, \ldots, n$. Let $s \in \{1, 2, 3\}$. Assume that $a_s < s$. Since $\sum_{i=1}^{n} a_i = 3n$ and $a_s < s$ it follows that there exist some $k \in \{s + 1, s + 2, \ldots, n\}$ such that $a_k > s$.

Define a point $a' = (a'_1, a'_2, \ldots, a'_n)$ by: $a'_i = \begin{cases} a_i, & i \neq s, k \\ s, & i = s \\ a_k + a_s - s, & i = k \end{cases}$. It follows that $\sum_{i=1}^{n} a'_i = \sum_{i=1}^{n} a_i = 3n$. Moreover, $0 < a_k + a_s - s < a_k$ implies that $a'_k < k$ and, consequently that $a' \in H_n$. On the other hand, from $a_s(a_k - s) < s(a_k - s)$ follows $a_s a_k < s(a_k + a_s - s) = a'_k a'_k$ and, consequently $\prod_{i=1}^{n} a_i < \prod_{i=1}^{n} a'_i$. This contradicts the choice of $a$. Thus $a_s = s$, $s = 1, 2, 3$. Then $a_4 + a_5 + \cdots + a_n = 3n - 6$. From the AM-GM inequality it follows that $\prod_{i=4}^{n} a_i \leq \left( \frac{3n - 6}{n - 3} \right)^{n-3}$ with equality for $a_i = \frac{3n - 6}{n - 3}$, $i \geq 4$.

Moreover since for $n \geq 6$, $\frac{3n - 6}{n - 3} \leq 4$ it follows that $\left( 1, 2, \frac{3n - 6}{n - 3}, \ldots, \frac{3n - 6}{n - 3} \right) \in H_n$.

Thus, the maximal value of $\prod_{i=1}^{n} a_i$ is $6 \cdot \left( \frac{3n - 6}{n - 3} \right)^{n-3}$ if $n \geq 6$ and 120 if $n = 5$.

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Three submitters assumed that the $a_i$ had to be integers, and if they solved this equivalently hard problem that solution was counted correct.
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