Solution 1 by Bennett Marsh, Purdue Junior, Physics/Math

By the definition of the limit, for any $\varepsilon > 0$ there exists some $K(\varepsilon)$ such that if $x > K(\varepsilon)$, then $|f(x) + f'(x) - A| < \varepsilon$. We now prove a few facts about $f(x)$;

1. There exists some $t > K(\varepsilon)$ such that $f(t) < A + \varepsilon$. Suppose otherwise. Then for all $x > K(\varepsilon/2)$, $f'(x) < A + \varepsilon/2 - f(x) < -\varepsilon/2$. But then eventually, $f(x)$ must cross over $A + \varepsilon$, contradicting the assumption.

2. For all $x > t$, $f(x) < A + \varepsilon$. Otherwise, by continuity, $f(x)$ must hit $A + \varepsilon$ from below with nonnegative slope, making $f(x) + f'(x) \geq A + \varepsilon$, a contradiction.

3. By flipping the signs in the above arguments, it can be seen that there exists some $s > K(\varepsilon)$ such that $f(s) > A - \varepsilon$ for all $x > s$.

4. Therefore, $|f(x) - A| \leq \varepsilon$ for all $x > \max(s, t)$.

Since $\varepsilon$ was arbitrary, this proves that $\lim_{x \to \infty} f(x) = A$.

Sketch of Solution 2: (A composite from several solvers)

L’Hopital and the fact that $\lim_{x \to \infty} \frac{g'(x)}{e^x} = A$ implies $\lim_{x \to \infty} \frac{g(x)}{e^x} = A$, applied to $g(x) = e^x f(x)$ (note $g'(x) = e^x (f(x) + f'(x))$) gives $\lim_{x \to \infty} f(x) = A$.

The problem was also solved by:

Yucheng Chen (Undergraduate, Fr. Engr.)
Suhas Sreehari (Graduate, ECE)
Marco Biagini (Italy)
Hongwei Chen (Professor, Christopher Newport Univ.)
Hubert Desprez (Paris, France)
Jon Dewitt (Student, Haverford College)
Sandipan Dey (UMBC Alumni)
David Elden (Purdue Alumni)
Tin Lam (Engineer, St. Louis, MO)
Steven Landy (Physics Faculty, IUPUI)
Wei-xiang Lien (Taiwan)
Adem Limani (Student, U of Lund, Sweden)
Perfetti Paolo (Roma, Italy)
M. Rajeswari (TA, Anna Univ., India)
Sorin Rubinstein (TAU staff, Tel Aviv, Israel)
Eduardo Escamilla Saldana (Mexico)
Craig Schroeder (Post doc, UCLA)
David Stigant
David Stoner
Justin Wolfe (Graduate student, Old Dominion U)