PROBLEM OF THE WEEK
Solution of Problem No. 3 (Spring 2014 Series)

Solution 1 by Bennett Marsh, Purdue Junior, Physics/Math

By the definition of the limit, for any $\varepsilon > 0$ there exists some $K(\varepsilon)$ such that if $x > K(\varepsilon)$, then $|f(x) + f'(x) - A| < \varepsilon$. We now prove a few facts about $f(x)$;

1. There exists some $t > K(\varepsilon)$ such that $f(t) < A + \varepsilon$. Suppose otherwise. Then for all $x > K(\varepsilon/2), f'(x) < A + \varepsilon/2 - f(x) < -\varepsilon/2$. But then eventually, $f(x)$ must cross over $A + \varepsilon$, contradicting the assumption.
2. For all $x > t, f(x) < A + \varepsilon$. Otherwise, by continuity, $f(x)$ must hit $A + \varepsilon$ from below with nonnegative slope, making $f(x) + f'(x) \geq A + \varepsilon$, a contradiction.
3. By flipping the signs in the above arguments, it can be seen that there exists some $s > K(\varepsilon)$ such that $f(x) > A - \varepsilon$ for all $x > s$.
4. Therefore, $|f(x) - A| \leq \varepsilon$ for all $x > \max(s, t)$.

Since ε was arbitrary, this proves that $\lim\limits_{x\to\infty} f(x) = A$.

Sketch of Solution 2: (A composite from several solvers)

L'Hopital and the fact that $\lim\limits_{x\to\infty} \dfrac{g'(x)}{e^x} = A$ implies $\lim\limits_{x\to\infty} \dfrac{g(x)}{e^x} = A$, applied to $g(x) = e^x f(x)$ (note $g'(x) = e^x(f(x) + f'(x))$) gives $\lim\limits_{x\to\infty} f(x) = A$.

The problem was also solved by:
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