PROBLEM OF THE WEEK
Solution of Problem No. 4 (Spring 2014 Series)

Problem:
Let \( p \) be a polynomial in the variables \( x_1, x_2, \ldots, x_n \). Show that if there is a number \( C \) such that \( |p(x_1, \ldots, x_n)| \leq C \) for all real \( x_1, x_2, \ldots, x_n \) then there is a number \( r \) such that \( p(x_1, \ldots, x_n) = r \) for all \( x_1, \ldots, x_n \).

Solution 1: (by Bennett Marsh, Physics/Math. Junior, Purdue University)

Assume that \( |p(\vec{x})| \leq C \) for all \( \vec{x} \in \mathbb{R}^n \), and that there exist \( \vec{x}_1, \vec{x}_2 \) such that \( p(\vec{x}_1) \neq p(\vec{x}_2) \). Define \( f(t) = p(\vec{x}_1 + (\vec{x}_2 - \vec{x}_1)t) \). Now \( f(t) \) is a polynomial in \( t \), and since \( f(0) \neq f(1) \), it is nonconstant, say of degree \( m > 0 \). Then letting \( f(t) = \sum_{k=0}^{m} a_k t^k \), we see that \( \lim_{t \to \infty} f(t)/t^m = a_m \neq 0 \). But this implies that \( \lim_{t \to \infty} f(t) = \pm \infty \), so \( f(t) \), and thus also \( p(\vec{x}) \), is unbounded. This contradicts the initial assumption, so \( p(\vec{x}) \) must in fact be constant.

Solution 2: (by Craig Schroeder, UCLA Postdoc)

Fix \( x_1, \ldots, x_n \) and let \( f(t) = p(x_1 t, \ldots, x_n t) \). Then \( f \) is a univariate polynomial which is bounded and so \( f(1) = f(0) \), i.e. \( p(x_1, x_2, \ldots, x_n) = p(0, 0, \ldots, 0) \). [Then use that bounded polynomials in one variable are bounded].

The problem was also solved by:

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