PROBLEM OF THE WEEK
Solution of Problem No. 5 (Spring 2014 Series)

Problem:

An urn contains n balls numbered $1, 2, \ldots, n$. They are drawn one at a time at random until the urn is empty. Find the probability that throughout this process the numbers on the balls which have been drawn is an interval of integers. [That is, for $1 \leq k \leq n$, after the kth draw the smallest number drawn equals the largest drawn minus $k-1$.]

Solution: (by Jason L. Smith, Professor, Richland Community College, IL)

Let S_k be the set of balls already drawn from the urn at step k that satisfy the conditions of the problem.

Instead of constructing various S_k from scratch, imagine "deconstructing," starting with $S_n = \{1, 2, 3, \ldots, n\}$. The number of possibilities for S_k at each stage of the reverse process is equal to the number of possibilities at that stage counting in the "forward" direction.

There are two allowed choices to obtain S_{n-1} from S_n: remove ball 1 or ball n from set S_n. At step k of deconstruction, there are likewise two allowed choices (out of a total of k choices) to obtain the next smaller set: remove either the largest or smallest member. When only one element remains (set S_1), there is just one choice about which ball to remove. The probability of having an acceptable set S_k at each stage is therefore

$$\frac{2}{n} \cdot \frac{2}{n-1} \cdot \frac{2}{n-2} \cdot \cdots \cdot \frac{2}{3} \cdot \frac{2}{2} \cdot \frac{1}{1} = \frac{2^{n-1}}{n!}.$$

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