PROBLEM OF THE WEEK
Solution of Problem No. 13 (Spring 2015 Series)

Problem:

Let Q^+ denote the positive rationals. Let f denote a function on Q^+ to Q^+ which satisfies the equation $f(u\,f(v)) = f(u)/v$ for all u and v in Q^+. Show that f must be $1-1$ and onto, and that $f(uv) = f(u)f(v)$. Also give an explicit example of such a function. Hint: first show that $f(1) = 1$.

<u>**Solution by Charles Burnette, Graduate Student, Drexel University**</u>

We first show that $f(1) = 1$. To see why, note that because $f(1) \in \mathbb{Q}^+$, we can take $f(f(1)) = f(1f(1)) = f(1)/1 = f(1)$. On the other hand $f(f(1)) = f(1f(f(1))) = f(1)/f(1) = 1$.

Now let $a, b \in \mathbb{Q}^+$ with $f(a) = f(b)$. Then $f(f(a)) = f(f(b))$. Since

$$f(f(x)) = f(1f(x)) = f(1)/x = 1/x$$

for all $x \in \mathbb{Q}^+$, it follows that $1/a = 1/b$, and so $a = b$. Hence f is $1-1$. Furthermore, as previously stated, $f(f(1/a)) = 1/(1/a) = a$. Thus f is onto. Lastly, because

$$f(1/x) = f(f(f(x))) = f(1f(f(x))) = f(1)/f(x) = 1/f(x)$$

for all $x \in \mathbb{Q}^+$, we have that

$$f(uv) = f(uf(f(1/v))) = f(u)/f(1/v) = f(u)/(1/f(v)) = f(u)f(v).$$

To construct an explicit example, it suffices to decide what the prime numbers should map to and then extend to $\mathbb{Q}^+$ using multiplicativity and the fact that $f(1/x) = 1/f(x)$. Indeed, if p_i denotes the i^{th} prime, then set $f(p_{2k-1}) = p_{2k}$ and set $f(p_{2k}) = 1/p_{2k-1}$. (In fact, any involution on the set of primes induces a map satisfying our conditions.)

NOTE: $f(x) = x^i$, where $i = \sqrt{-1}$, is not an example because, for example, $f(2) = 2^i = \cos(\ln 2) + i\sin(\ln 2)$ is not rational.
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