PROBLEM OF THE WEEK

Solution of Problem No. 2 (Spring 2015 Series)

Problem:

Let f be a real valued function defined on $D = \{(i,j) : i$ and j are integers in $[-2015, 2015]\}$ such that

$$f(i,j) = (1/4)[f(i+1,j) + f(i-1,j) + f(i,j+1) + f(i,j-1)]$$

when i and j are both in $(-2015, 2015)$ and $f(i,j) = 0$ if one or both of $|i|, |j|$ is 2015. Prove $f(i,j) = 0$ for all (i,j) in D.

Solution by Victor Lee, Computer Science at Purdue

Since the image of f is a finite set, f must attain max and min in D. Suppose $f(i,j) = M$ is max. Assume $|i| < 2015$ and $|j| < 2015$, otherwise, $M = 0$. $f(i-1,j), f(i+1,j), f(i,j-1), f(i,j+1)$ are all less than or equal to M, but their average is $f(i,j) = M$, so $f(i-1,j) = f(i+1,j) = f(i,j-1) = f(i,j+1) = M$. Continue this method to $f(i-1,j)$, we get $f(i-2,j) = M$. Repeat this, we will get $M = 0$ as we touch the side of D. Therefore, $f \leq 0$. By the same argument, min is also zero, so $f \geq 0$. So, $f = 0$ on D.

The problem was also solved by:

Undergraduates: Victor Lim (CS), Bennett Marsh (Sr. Physics & Math), Jiaqi Zhou (Math)

Graduates: Tairan Yuwen (Chemistry)

Others: Chao-yi Chang (Graduate Student, Taiwan), Hongwei Chen (Professor, Christopher Newport Univ. Virginia), Jiehua Chen (Quantitative Engineering Design Inc.), Mark Crawford Jr. (Professor, Waubonsee Community College, IL), Hubert Desprez (Paris, France), Sean Dillard (Duke U), Steven Landy (Physics Faculty, IUPUI), Matthew Lim, F. Ogane, Thomas M Roddenberry (HS Student, FL), Sorin Rubinstein (TAU faculty, Tel Aviv, Israel), Craig Schroeder (Postdoc. UCLA), Feng Tian (Studnet, Duke U)