PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2000 Series)

Problem: Given a triangle with vertices A, B, C and points A_1, B_1, C_1 on the sides $\overline{BC}, \overline{CA}$ and $\overline{AB}$, respectively, prove that the circumcircles of the triangles $\triangle AB_1C_1$, $\triangle BA_1C_1$, and $\triangle CA_1B_1$ have a common point.

Solution (by the Panel)

The circumcircles of $\triangle AB_1C_1$ and $\triangle BA_1C_1$ have the point C_1 in common, hence have another point P in common unless they are tangent (to be discussed later). There are two cases to be considered.

a) P lies inside $\triangle ABC$, then we have quadrangles AB_1C_1P and BPA_1C_1 inscribed in the circles. It follows that $\angle B_1PC_1 = 180 - \angle B_1AC_1$ and $\angle A_1PC_1 = 180 - \angle A_1BC_1$. So $\angle B_1PA_1 = 180 - \angle A_1CB_1$; thus the quadrangle B_1PA_1C has a circumcircle and P lies on the circumcircle of $\triangle B_1CA_1$.

b) If any pair of the circumcircles intersects in a point other than A_1, B_1, or C_1, relabel the original triangle so these are the circumcircles of $\triangle AB_1C_1$ and $\triangle BA_1C_1$. Now the quadrangles AB_1C_1P and BPA_1C_1 are not convex, and $\angle B_1PC_1 = \angle B_1AC_1$ and $\angle A_1PC_1 = \angle A_1BC_1$. The quadrangle CB_1PA_1 is convex and $\angle B_1PC_1 + \angle A_1PC_1 = 180 - B_1CA_1$; therefore this quadrangle has a circumcircle which must be that of $\triangle B_1CA_1$, so P lies on this circle.

c) If two of the circumcircles are tangent, say at point C_1, then C_1 is a limit point of points for which such tangency does not occur, and the result is obtained by continuity.

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