PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2000 Series)

**Problem:** Evaluate the integral $I = \int_0^\infty \frac{\text{Arctan}(ax) - \text{Arctan}(bx)}{x}\,dx$ where $a$ and $b$ are positive numbers. Hint: Express $I$ as a double integral.

**Solution** (by Gajath Gunatillake, Gr. Math, and many others)

$$\text{Arctan}\, ax - \text{Arctan}\, bx = \int_b^a \frac{x}{1 + x^2t^2}\,dt.$$

Hence,

$$I = \int_0^\infty \frac{1}{x} \int_b^a \frac{x}{1+x^2t^2}\,dt\,dx = \int_0^\infty \int_b^a \frac{1}{1+x^2t^2}\,dt\,dx.$$

Change of order of integration gives

$$\begin{aligned} I &= \int_b^a \int_0^\infty \frac{1}{1+x^2t^2}\,dx\,dt = \int_b^a \frac{1}{t} \int_0^\infty \frac{t}{1+x^2t^2}\,dx\,dt \\ &= \int_b^a \frac{1}{t}\left(\frac{\pi}{2} - 0\right)dt = \frac{\pi}{2}\log\left(\frac{a}{b}\right). \end{aligned}$$

Also solved by:

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Graduates: Vikram Buddhi (MA), Yi-Ru Huang (Stat), Wook Kim (MA), Sravanthi Konduri (CE), B. N. Reddy Vanga (Nucl E)

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There was one incorrect solution.

**Correction**: A correction must be made to the published solution of Problem No. 5, as pointed out by Vikram Buddhi.

$$P_n(x) > x\frac{x^{2n+1}+1}{x+1} + 2n + 1$$

should be replaced by

$$(1+x)P_n(x) = x\frac{x^{2n+1}+1}{x+1} + 2n + 1, \text{ hence } P_n(x) > 0 \text{ for } x > 0.$$