PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2001 Series)

Problem: Determine all the square integers whose decimal representations end in 2001. What is the smallest of these numbers?

Solution (by Mike Hamburg, Sr. St. Joseph H.S., South Bend)

We seek $n^2 \equiv 2001 \pmod{10}^4 . n = 1001$ is obviously a solution, so $n^2 \equiv 1001^2, n^2 - 1001^2 \equiv 0$, so $(n+1001)(n-1001) \equiv 0$ (all mod 10^4). $10^4 = 2^4 \cdot 5^4$, so $2^4|(n+1001)(n-1001)$. Although 2 can divide both $n+1001$ and $n-1001$, 4 cannot divide them both because they differ by 2002. Similarly, $5^4|(n+1001)(n-1001)$ and since 5 cannot divide them both, $5^4|(n+1001)$ or $5^4|(n-1001)$. We also have $8|(n+1001)$ or $8|(n-1001)$. Reducing mod $5^4 = 625$ and 8, we have $n \equiv \pm 1 \pmod{8}$ and $n \equiv \pm 249 \pmod{5^4}$. Since $625 \equiv 1 \pmod{8}$ and 8 is inverse to 547 (mod 625), the Chinese Remainder Theorem gives us $n \equiv (\pm 1) \cdot 625 + (\pm 249) \cdot 8 \cdot 547 \pmod{5^4 \cdot 8 = 5000}$. Reducing mod 5000, we have $n \equiv 249, 1001, 3999$ or $4751 \pmod{5000}$. We check that the squares of these numbers end in 2001 and that $(n + k5000)^2 = k^2 5000^2 + 10000kn + n^2 \equiv n^2 \pmod{5000}$.

Also solved (at least partially) by:

Undergraduates: Jim Hill (Jr. MA), Piti Irawan (Sr. CS/MA), Aftab Mohammed Jalal (So. CS/MA), Stevie Schrauder (Sr. CS/MA), Eric Tkaczyk (Jr. EE/MA)

Graduates: Rajender Adibhatla (MA), John Hunter (MA), Chris Lomont (MA), K. H. Sarma (Nuc E), Amit Shirsat (CS), P. Ghosh & D. Subramanian (CHME)

Faculty & Staff: Steven Landy (Phys. at IUPUI), Chris Maxwell (OB & FC, Purdue)

Others: Jonathan Landy (Warren Central H.S., Indpls), Jason VanBilliard (Fac. Phila. Biblical Univ. Langhorne, PA), Aditya Utturwar (Grad. AE, Georgia Tech)

One unacceptable solution was received.