PROBLEM OF THE WEEK

Solution of Problem No. 5 (Fall 2001 Series)

Problem: Suppose $a, b \in \mathbb{C}$ (complex numbers) and $b \neq 0$. Let $(x^2+ax+b)^{-1} = \sum_{k=0}^{\infty} c_k x^k$ for $|x|$ sufficiently small. Show that the ratio of determinants

$$\begin{vmatrix} c_k & c_{k+1} \\ c_{k+1} & c_{k+2} \end{vmatrix} \Big/ \begin{vmatrix} c_{k+1} & c_{k+2} \\ c_{k+2} & c_{k+3} \end{vmatrix}$$

is independent of k.

Solution (by Eric Tkaczyk, Jr. EE and MA)

Since we are given $1 + 0x + 0x^2 + \cdots = (x^2+ax+b)\sum_{k=0}^{\infty} c_k x^k$, the following recurrence relation must hold:

$$bc_{k+2} + ac_{k+1} + c_k = 0, \quad \forall k \in \mathbb{N}.$$

Whence

(1) $$c_k = -ac_{k+1} - bc_{k+2}$$

and

(2) $$bc_{k+3} = -ac_{k+2} - c_{k+1}.$$

Thus

$$\begin{aligned} c_k c_{k+2} - c_{k+1}^2 &= c_{k+1}(-ac_{k+2} - c_{k+1}) - bc_{k+2}^2, \quad \text{by (1)} \\ &= c_{k+1}(bc_{k+3}) - bc_{k+2}^2, \quad \text{by (2)} \\ &= b(c_{k+1}c_{k+3} - c_{k+2}^2). \end{aligned}$$

Hence, the ratio of the determinants is b, clearly invariant with respect to k.

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