PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2001 Series)

**Problem:** Evaluate $\lim_{x\to 0} \dfrac{\sin(\text{Arctan } x) - \tan(\text{Arcsin } x)}{\text{Arcsin}(\tan x) - \text{Arctan}(\sin x)}$.

**Solution** (by Steven Landy, Fac. Physics at IUPUI)

The Taylor series of these functions are

$$\sin x = x - \frac{x^3}{6} + 0(x^5) \qquad \sin^{-1}(x) = x + \frac{x^3}{6} + 0(x^5)$$
$$\tan x = x + \frac{x^3}{3} + 0(x^5) \qquad \tan^{-1}(x) = x - \frac{x^3}{3} + 0(x^5).$$

Substituting these into our expression gives

$$\frac{\left[(x - \frac{x^3}{3}) - \frac{(x-\frac{x^3}{3})^3}{6}\right] - \left[(x + \frac{x^3}{6}) + \frac{(x+\frac{x^3}{6})^3}{3}\right] + 0(x^5)}{\left[(x + \frac{x^3}{3}) + \frac{(x+\frac{x^3}{3})^3}{6}\right] - \left[(x - \frac{x^3}{6}) - \frac{(x-\frac{x^3}{6})^3}{3}\right] + 0(x^5)}$$

$$= \frac{x - \frac{x^3}{3} - \frac{x^3}{6} - x - \frac{x^3}{6} - \frac{x^3}{3} + 0(x^5)}{x + \frac{x^3}{3} + \frac{x^3}{6} - x + \frac{x^3}{6} + \frac{x^3}{3} + 0(x^5)} = \frac{-x^3 + 0(x^5)}{x^3 + 0(x^5)}.$$

Thus the limit as $x \to 0$ is $-1$.

Also solved by:

Undergraduates: Haizhi Lin (Jr. MA)

Graduates: Ali R. Butt (ECE), Danlei Chen (CHME), D. Subramanian & P. Ghosh (CHME)

Others: Michael Hamburg (Sr. St. Joseph's H.S., South Bend), Dan Vanderkam (Sr. St. Joseph's H.S., South Bend)

Five unacceptable solutions were received.

Remark. It is unacceptable to replace e.g. $\sin(\text{Arc tan } x)$ by $\sin x$ in this problem.