PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2001 Series)

Problem: Let the sequence $\{x_n\}$ of integers (modulo 11) be defined by the recurrence relation $x_{n+3} \equiv \frac{1}{3}(x_{n+2} + x_{n+1} + x_n) \pmod{11}$ for $n = 1, 2, \cdots$. Show that every such sequence $\{x_n\}$ is either constant or periodic with period 10.

Solution (by the Panel)

The general solution of a 3-term recurrence relation is a linear combination of 3 linearly independent solutions. Three linearly independent solutions (which can be arrived at by the use of the characteristic equation $3r^3 - r^2 - r - 1 \equiv 0$) are:

$$x_n \equiv 1^n, \quad x_n \equiv (-3)^n, \quad x_n \equiv (-5)^n \qquad \pmod{11}.$$

So the general solution is

$$x_n \equiv A + B(-3)^n + C(-5)^n \qquad \pmod{11}.$$

Now $(-3)^{10} \equiv 1,\quad$ while $(-3)^2 \not\equiv 1, \quad (-3)^5 \not\equiv 1,$

and also $(-5)^{10} \equiv 1, \quad$ while $(-5)^2 \not\equiv 1, \quad (-5)^5 \not\equiv 1.$

So $x_n \equiv A$ if $B = C = 0$, while $\{x_n\}$ is constant or $\{x_n\} = \{A + B(-3)^n + C(-5)^n\}$ if $BC \neq 0$, which has period 10.

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