PROBLEM OF THE WEEK
Solution of Problem No. 6 (Fall 2002 Series)

Problem: Suppose $f(x)$ is a polynomial with integer coefficients and degree $n \geq 2$, and suppose $|f(x_i)|$ is prime for at least $2n+1$ integers x_i. Show that:

a) $f(x)$ is irreducible, that is, $f(x)$ is not the product of two polynomials of degree ≥ 1 with integer coefficients.

b) for at least one value of n, (a) does not hold if $2n+1$ is replaced by $2n$.

Solution (by Eric Tkaczyk, Sr. EE/MA)

Proof:

a) Assume, conversely, that $f(x) = g(x)h(x)$, where $g(x)$, $h(x)$ are polynomials of degree m and k respectively ≥ 1, with integer coefficients and $m + k = n$. Now, the polynomials $g(x)+1$, $g(x)-1$, $h(x)+1$, and $h(x)-1$ can have at most m, m, k, and k distinct integer roots, respectively. So there are at most $m + m + k + k = 2n$ x_i's for which $|g(x_i)|$ or $|h(x_i)| = 1$. Thus, if $f(x)$ is reducible, $|f(x)|$ will be prime for at most $2n$ integers x_i. This proves (a).

b) As a counterexample for the case $n = 2$, consider $f(x) = (2x+1)(x-2)$. Clearly, $f(x)$ is reducible, and $|f(x)|$ is prime for x in $\{-1, 0, 1, 3\}$. So (a) does not hold if $2n+1$ is replaced by $2n$.

Also solved by:

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J.L.C. (Fishers, IN) submitted a correct solution of Problem 5 which, though late, we have credited to him.