PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2002 Series)

Problem: Let f be a function $\mathbb{R} \to \mathbb{R}$ which is n times differentiable. Determine the coefficient of h^n in the Taylor expansion of $f((x+h)^2)$. (The answer should be in the form $\sum_k f^{(k)}(x^2)P_k(x)$ where $P_k(x)$ is a monomial.)

Solution (by Yifan Liang, Gr. ECE, edited by the Panel)

The Taylor expansion of $f((x+h)^2)$ at x^2 is

$$f((x+h)^2) = f(x^2 + h(h+2x)) = \sum_{k=0}^{n} \frac{f^{(k)}(x^2)}{k!} h^k (h+2x)^k + o(h^n).$$

Only items with $[\frac{n}{2}] \leq k \leq n$ contribute to h^n, while also

$$(h+2x)^k = \sum_{i=0}^{k} \binom{k}{i} (2x)^{k-i}\, h^i.$$

Let $i = n-k \geq 0$, $k-i = 2k-n \geq 0$. The coefficient is

$$\sum_{k=[\frac{n}{2}]}^{n} \frac{f^{(k)}(x^2)}{k!} \binom{k}{n-k} (2x)^{2k-n} = \sum_{k=[\frac{n}{2}]}^{n} f^{(k)}(x^2) P_k(x)$$

where $P_k(x) = \dfrac{1}{k!}\dbinom{k}{n-k}(2x)^{2k-n} = \dfrac{(2x)^{2k-n}}{(n-k)!(2k-n)!}$, $[\dfrac{n}{2}] \leq k \leq n$.

Also solved by:

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One incorrect solution was received.