PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2002 Series)

Problem: Find the integers n for which $S = 2^{1994} + 2^{1998} + 2^{1999} + 2^{2000} + 2^{2002} + 2^n$ is a perfect square.

Solution (by Sharmashankar Subramanian, Gr. ChE, edited by the Panel)

$$\begin{aligned} S &= 2^{1994}(1 + 2^4 + 2^5 + 2^6 + 2^8 + 2^{n-1994}) \\ &= 2^{1994}(369 + 2^m), \text{ where } m = n - 1994. \end{aligned}$$

Since 2^{1994} is a square, S is a square if and only if $369 + 2^m$ is a square, $369 + 2^m = b^2$, b an integer. If m is odd, say $m = 2k + 1$, then $369 + 2 \cdot 2^{2k} = b^2$ and $0 - 1 \equiv b^2 \pmod 3$, which is impossible. Hence m is even, and we have

$$369 = (b - 2^{\frac{m}{2}})(b + 2^{\frac{m}{2}}), \text{ so that}$$

each of $(b - 2^{\frac{m}{2}})$ and $(b + 2^{\frac{m}{2}})$ must be a divisor of $369 = 1 \times 369 = 3 \times 123 = 9 \times 41$. There are only three cases:

$$\begin{aligned} &b - 2^{\frac{m}{2}} = 1,\ b + 2^{\frac{m}{2}} = 369 \Rightarrow 2^{\frac{m}{2}} = 184, \text{ not possible;} \\ &b - 2^{\frac{m}{2}} = 3,\ b + 2^{\frac{m}{2}} = 123 \Rightarrow 2^{\frac{m}{2}} = 60, \text{ not possible;} \\ &b - 2^{\frac{m}{2}} = 9,\ b + 2^{\frac{m}{2}} = 41 \Rightarrow 2^{\frac{m}{2}} = 16, m = 8. \end{aligned}$$

Thus $m = 8$, $n = 2002$ is the only solution.

Also solved by:

Graduates: Yifan Liang (EE), Maddipati Sridhar (ChE),

Faculty: Steven Landy (Physics at IUPUI)

Others: J.L.C. (Fishers, IN), Vijay Madhavapeddi (Newark, CA), Namig Mammadov (Baku, Azerbaijan), Vivek Mehra (Mumbai U., India), Peter Montgomery (San Rafael, CA), Steve Taylor (Middletown H.S., OH), Alex Miller (St. Anthony H.S., MN)

Remarks Several solvers started with the unjustified assumption that S is the square of a sum of three powers of 2. They received reduced credit. We did not list Steven Landy (Physics, IUPUI) as a solver of Problem 12. We have corrected this in our records.