PROBLEM OF THE WEEK

Solution of Problem No. 2 (Fall 2003 Series)

Problem: A real-valued function $f(x)$ has a continuous second order derivative $f''(x) > 0$ on $a < x < b$. It is to be approximated by a linear function $\ell(x) \leq f(x)$ so that $\int_a^b (f(x) - \ell(x))dx$ is minimal. Determine $\ell(x)$.

Solution (by Rob Pratt, U. North Carolina, Chapel Hill, NC)

It is clear that $\ell(x) = f(x)$ for at least one value of x since otherwise we can shift the graph of ℓ up, reducing the value of the integral. Hence the graph of ℓ is tangent to the graph of f, and therefore $\ell(x) = f(c) + f'(c)(x - c)$ for some c in (a, b). Note that minimizing

$$\int_a^b (f(x) - \ell(x))dx = \int_a^b f(x)dx - \int_a^b \ell(x)dx$$

is equivalent to maximizing

$$\begin{aligned}\int_a^b \ell(x)dx &= \int_a^b (f'(c)x + f(c) - cf'(c))dx \\ &= \frac{f'(c)(b^2 - a^2)}{2} + (f(c) - cf'(c))(b - a) \\ &= (b - a)\left(\frac{f'(c)(a + b)}{2} + f(c) - cf'(c)\right),\end{aligned}$$

which is equivalent to maximizing $g(c) = f'(c)(a + b)/2 + f(c) - cf'(c)$. Now

$$g'(c) = f''(c)(a + b)/2 + f'(c) - cf''(c) - f'(c) = f''(c)((a + b)/2 - c).$$

Since $f''(x) > 0$, we have $g'(c) > 0$ for $c < (a + b)/2, g'(c) = 0$ for $c = (a + b)/2$, and $g'(c) < 0$ for $c > (a + b)/2$. Hence g is maximized when $c = (a + b)/2$, and so

$$\ell(x) = f\left(\frac{a + b}{2}\right) + f'\left(\frac{a + b}{2}\right)\left(x - \frac{a + b}{2}\right).$$

The solution is the line tangent to the curve at the midpoint of the interval (a, b).

Also solved by:

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Five incorrect solutions were received.