PROBLEM OF THE WEEK

Solution of Problem No. 5 (Fall 2003 Series)

Problem: The triangle $\triangle$ has angles α, β, γ opposite respectively to the sides a, b, c. Show that $\triangle$ is equilateral if and only if $ab\cos\gamma = ac\cos\beta = bc\cos\alpha$.

Solution (by Chad Aeschliman, Soph. ECE)

By the law of cosines,

$$\cos\gamma = (a^2 + b^2 - c^2)/2ab,$$
$$\cos\beta = (a^2 - b^2 + c^2)/2ac,$$
$$\cos\alpha = (-a^2 + b^2 + c^2)/2bc.$$

Substituting these into the given equation yields $a^2+b^2-c^2 = a^2-b^2+c^2 = -a^2+b^2+c^2$. Looking at the first equality we get $b^2 = c^2$, and looking at the last equality we get $a^2 = b^2$.

Thus $a^2 = b^2 = c^2$, or $|a| = |b| = |c|$ which is the definition of an equilateral triangle. The converse conclusion is trivial.

Another solution (by the Panel).

Let $\overline{a}, \overline{b}, \overline{c}$ denote the vectors from B to C, C to A, A to B, resp. Then (1) $\overline{a} + \overline{b} + \overline{c} = 0$. Given is $\overline{a}\cdot\overline{b} = \overline{b}\cdot\overline{c} = \overline{c}\cdot\overline{a}$ (inner products). Hence $\overline{b}\cdot(\overline{a}-\overline{c}) = 0$ and by (1) $(\overline{a}+\overline{c})(\overline{a}-\overline{c}) = \overline{a}^2 - \overline{c}^2 = 0$, hence $a = c$, likewise $c = b$. Hence $a = b = c$.

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