PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2003 Series)

Problem: For a given positive integer n, evaluate

$$\int_0^{2\pi} \cos x \cdot \cos 2x \cdot \cos 4x \cdots \cos 2^{n-1}x \cdot \cos(2^n - 1)x\, dx. \tag{1}$$

Solution (by Fijoy George, Grad. CS, edited by the Panel)

We have the equation

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}. \tag{2}$$

Thus, (1) can be rewritten as

$$\frac{1}{2^{n+1}} \int_0^{2\pi} (e^{ix} + e^{-ix})(e^{2ix} + e^{-2ix}) \cdots (e^{2^{n-1}ix} + e^{-2^{n-1}ix})(e^{(2^n-1)ix} + e^{-(2^n-1)iz})dx. \tag{3}$$

Now, multiplying the brackets in (3), we get a sum of terms e^{imx} with $m \neq 0$ except two terms e^{0ix}. Using the equations

$$\int_0^{2\pi} e^{imx} dx = \begin{cases} 2\pi \text{ if } m = 0 \\ 0 \quad \text{otherwise} \end{cases}, \tag{4}$$

we obtain

$$(3) = \frac{1}{2^{n+1}}(2\pi + 2\pi) = \frac{\pi}{2^{n-1}}.$$

Thus,

$$\int_0^{2\pi} \cos x \cdot \cos 2x \cdot \cos 4x \cdots \cos 2^{n-1}x \cdot \cos(2^n - 1)x\, dx = \frac{\pi}{2^{n-1}}.$$

Also solved by:

Undergraduates: Chad Aeschliman (So. ECE), Michael Chun Chang (So. Bio/Chem), Kedar Hippalgaonkar (Fr. ME), Jignesh V. Mehta (So. Phys)

Graduates: Ali R. Butt (ECE), Tom Engelsman (ECE), Jianguang Guo (Phys), Ankur Jain (ChE), Yifan Liang (ECE), K. H. Sarma (NucE), Kshitij Shrotri (AAE), Brahma N.R. Vanga (Nucl.)

Faculty: Steven Landy (Physics at IUPUI)

Others: Prithwijit De (U.C.C. Cork, Ireland), Alex Miller (Fr. U. Minn.), Dr. Troy Siemers (MA, V.M.I.), Benjamin K. Tsai (NIST), Ram Venkatachalam (Murex)

Three incorrect solutions were received.

Several solutions of Problem 5 arrived late. We will credit them:
Undergraduates: Jason Arema (Jr. MA), Chris Carlevato (Sr.)
Graduates: Jianguang Guo (Phys), Yifan Liang (ECE), K. H. Sarma (Nucl), Kshitij Shrotri (AAE)
Faculty: Steven Landy (Phys at IUPUI)
Others: Jaypradesh Chipalkatti (Vancouver B.C.), Namig Mammadov (Azerbaijan), Dr. Troy Siemers (MA, VMI)
One anonymous solution of Problem 5 was received.