PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2003 Series)

Problem: Suppose a pond contains $x(t)$ fish at time t and $x(t)$ changes according to $\frac{dx}{dt} = x(1 - \frac{x}{x_0}) - f$, where x_0 is the equilibrium amount with no fishing and $f > 0$ is the constant rate of removal due to fishing. Assume $x(0) = \frac{x_0}{2}$.

a) If $f < \frac{x_0}{4}$, solve for $x(t)$ and show that it tends to an equilibrium amount between $\frac{x_0}{2}$ and x_0.

b) What happens if $f \geq \frac{x_0}{4}$?

Solution (by the Panel)

$$\frac{dx}{dt} = x - \frac{x^2}{x_0} - f = -\frac{1}{x_0}(x - \frac{x_0}{2})^2 + (\frac{x_0}{4} - f).$$

Let $\frac{1}{\sqrt{x_0}}(x - \frac{x_0}{2}) = y(t)$, so that $\sqrt{x_0}\, dy = dx$ and $y(0) = 0$.

Also let $a^2 = |\frac{x_0}{4} - f|$. In these terms the D.E. is

$$-dt = \frac{\sqrt{x_0}\, dy}{y^2 \mp a^2}$$

where $a = 0$ if $f = \frac{x_0}{4}$, negative if $f < \frac{x_0}{4}$, and positive if $f > \frac{x_0}{4}$.

(a) When $f < \frac{x_0}{4}$, $-t = \frac{\sqrt{x_0}}{2a} \log \frac{a-y}{a+y} + c$. Since $y(0) = 0$, we have $c = 0$, and so $e^{\frac{-2at}{\sqrt{x_0}}} = \frac{a-y}{a+y} = \frac{2a}{a+y} - 1$, i.e. $y = \frac{2a}{1+e^{\frac{-2at}{\sqrt{x_0}}}} - a$.

As $t \to \infty$, clearly $y \to a$, i.e. $\frac{1}{\sqrt{x_0}}(x - \frac{x_0}{2}) \to \sqrt{\frac{x_0}{4} - f}$, and so $x \to \frac{x_0}{2} + \sqrt{\frac{x_0^2}{4} - fx_0}$.

(b) When $f = \frac{x_0}{4}$, the original equation is $\frac{dx}{dt} = -(x - \frac{x_0}{2})^2/x_0$, which has the obvious constant solution $x(t) = \frac{x_0}{2} = x(0)$. When $f > \frac{x_0}{4}$, $-t = \frac{\sqrt{x_0}}{a} \arctan \frac{y}{a} + c$, where again $c = 0$. Now $-\tan \frac{at}{\sqrt{x_0}} = \frac{y}{a}$ or $-\sqrt{f - \frac{x_0}{4}} \tan \frac{\sqrt{f - \frac{x_0}{4}}}{\sqrt{x_0}}\, t = \frac{1}{\sqrt{x_0}}(x - \frac{x_0}{2})$, and so $x = \frac{x_0}{2} - \sqrt{fx_0 - \frac{x_0^2}{4}} \tan \sqrt{\frac{f}{x_0} - \frac{1}{4}}\, t$.

This is a decreasing function of t which becomes 0 when $\tan \sqrt{\frac{f}{x_0} - \frac{1}{4}}\, t = \frac{x_0}{2} \frac{1}{\sqrt{fx_0 - \frac{x_0^2}{4}}}$.

So the fish population becomes 0 in a finite time.

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Two incorrect solutions were received.