PROBLEM OF THE WEEK

Solution of Problem No. 9 (Fall 2003 Series)

Problem: Let S be the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \cdots$, and, for any fixed positive integer r, let R be the rearrangement of S into a series of groups of $2r$ positive terms followed by r negative terms. Determine the sum of R. (For example when $r = 1$, R is $(1 + \frac{1}{5} - \frac{1}{3}) + (\frac{1}{9} + \frac{1}{13} - \frac{1}{7}) + \cdots)$. You may use the fact that the sum of S is $\pi/4$.

Solution (by the Panel)

Let R_n be the sum of the first n terms of R. Then $R_{3rk} = S_{2rk} + \displaystyle\sum_{j=rk}^{2rk-1} \frac{1}{4j+1}$.

Now $\displaystyle\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{x}\, dx \right)$ exists, hence

$$\lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{1}{4j+1} - \int_1^n \frac{1}{4x+1}\, dx \right)$$

exists, hence

$$\lim_{k\to\infty} \left(\sum_{j=rk}^{2rk-1} \frac{1}{4j+1} - \int_{rk}^{2rk-1} \frac{1}{4x+1}\, dx \right) = 0,$$

thus

$$\lim_{k\to\infty} \sum_{j=rk}^{2rk-1} \frac{1}{4j+1} = \lim_{k\to\infty} \int_{rk}^{2rk-1} \frac{1}{4x+1}\, dx = \lim_{k\to\infty} \frac{1}{4} \log \frac{8rk-3}{4rk+1} = \frac{1}{4} \log 2.$$

Therefore,

$$R = \lim_{k\to\infty} R_{3rk} = S + \frac{1}{4} \log 2 = \frac{\pi}{4} + \frac{1}{4} \log 2.$$

Solved by:

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There were two incorrect solutions.