PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2003 Series)

Problem: Given n points $P_0, P_1, \cdots, P_{n-1} (n \geq 3)$, equally spaced on the unit circle. Determine $\sum_{0 \leq k < \ell < n} |P_k P_\ell|^2$.

Solution (by Steven Landy, Fac. Physics at IUPUI); edited by the Panel.

$$\begin{aligned} S &= \sum_{(0 \leq k < \ell < n)} |P_k P_\ell|^2 = \sum_{(0 < \ell < n)} \sum_{(0 \leq k < \ell)} |P_k P_\ell|^2 = \sum_{(0 < \ell < n)} \sum_{(0 \leq k < \ell)} |P_0 P_{\ell - k}|^2 \\ &= \sum_{(0 < \ell < n)} \sum_{(0 < j \leq \ell)} |P_0 P_j|^2. \end{aligned}$$

(Since $|P_0 P_{n-j}| = |P_0 P_j|$ and $|P_0 P_0| = 0$.)

$$S = (1/2) \sum_{(0 \leq \ell < n)} \sum_{(0 \leq j < n)} |P_0 P_j|^2 = (n/2) \sum_{(0 \leq \ell < n)} |P_0 P_\ell|^2.$$

Since

$$|P_0 P_\ell|^2 = (1 - e^{2\pi i (\ell/n)}) \cdot (1 - e^{-2\pi i (\ell/n)}) = 2 - e^{2\pi i (\ell/n)} - e^{-2\pi i (\ell/n)},$$

$$S = (n/2) \sum_{(0 \leq \ell < n)} (2 - e^{2\pi i (\ell/n)} - e^{-2\pi i (\ell/n)}) = n/2 \cdot 2n = n^2.$$

Also solved by:

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