PROBLEM OF THE WEEK

Solution of Problem No. 14 (Fall 2003 Series)

Problem: Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} (n = 1, 2, \cdots)$. Prove that $S_n - S_m$ is never an integer for $m < n$.

(Hint: for any k, between k and $2k$ there is at least one prime number.)

Solution (by Kedar Hippalgaonkar, Fr. ME; this is his second solution, which makes no use of the given hint; it is edited by the panel)

$$S_n - S_m = \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n}.$$

If $S_n - S_m$ is an integer then $S_n - S_m \geq 1$ hence $n \geq 2m$. There is a largest power of 2, say 2^k between $(m+1)$ and n, because if 2^{k-1} is largest power of 2 less than $(m+1)$ then 2^k is between $(m+1)$ and n.

The LCM $(m+1, \ldots, n) = 2^k 3^\ell 5^r \cdots$. So

$$S_n - S_m = \frac{(2^k 3^\ell \cdots)/(m+1)}{LCM} + \frac{(2^k 3^\ell \cdots)/(m+2)}{LCM} + \cdots \frac{(2^k 3^\ell \cdots)/2^k}{LCM} \cdots + \frac{(2^k 3^\ell)/n}{LCM}.$$

All the numerators are divisible by 2, except one, hence the sum is odd, while denominator is even. $S_n - S_m$ is not an integer.

Also solved by:

<u>Undergraduates</u>: Jason Anema (Jr. MA), Jignesh V. Mehta (So. Phys)

<u>Graduates</u>: Jianguang Guo (Phys)

<u>Faculty</u>: Steven Landy (Physics at IUPUI)

<u>Others</u>: Georges Ghosn (Quebec), Andrew Klein (Omaha), Chris Lomont (Cybernet, Ann Arbor, MI), Namig Mammadov (Baku, Azerbaijan), Angel Plaza (ULPGC Spain)

Two incorrect solutions were received.

Late solutions for Problem 13 from graduate students Jianguang Guo (Physics) and Gaurav Sharma (ECE).