PROBLEM OF THE WEEK

Solution of Problem No. 1 (Fall 2004 Series)

Problem: Determine the last two decimal digits of the numbers

$$A = 7 \cdot 7 \cdot 7 \cdots 7, \quad B = 7^{7^{7^{\cdot^{\cdot^{\cdot^{7}}}}}}$$

There are 2004 sevens in both A and B.

Solution (by Georges Ghosn, Quebec)

Using the congruence modulo 100:

$$7^4 \equiv 1 \pmod{100} \Rightarrow A = 7^{2004} = (7^4)^{501} \equiv 1 \pmod{100}$$

so the last two digits of A are 01.

$$B = 7^{7^{\cdot^{\cdot^{\cdot^{7}}}}} = 7^c$$

since $7^4 \equiv 1 \pmod{100}$, in order to know the congruence of B modulo 100 we have to find the congruence of C modulo 4.

But $7^2 \equiv 1 \pmod{4}$ and $c = 7^D$ (D is odd number) implies

$$\begin{aligned} c = 7^{2E+1} &\Rightarrow c \equiv 3 \pmod{4} \\ &\Rightarrow c = 4k + 3 \end{aligned}$$

hence $B = 7^{4k+3} \Rightarrow B = (7^4)^k \cdot 7^3 \equiv 7^3 \equiv 43 \pmod{100}$ so the last two digits of B are 43.

Also, at least partially solved by:

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Faculty: Steven Landy (Phys, IUPUI)

Others: George Barnett (Woodland CC, CA), Sanjiv (ECE, Waterloo)

There were 16 unacceptable solutions.

Comment: The correct answer, when arrived by faulty reasoning, is not an acceptable solution.