PROBLEM OF THE WEEK
Solution of Problem No. 3 (Fall 2004 Series)

Problem: Find all functions $f(x)$ defined for all real x and satisfying the equation

$$x\,f(y) + y\,f(x) = (x+y)f(x)f(y)$$

for all x and y. Prove that only two such f are continuous.

Solution (by Zachary Catlin, Fr. Mathematics, revised by the Panel)

Let us first consider the case when $y = x$. In this case, the equation becomes

$$x\,f(x) + x\,f(x) = (x+x)f(x)f(x)$$
$$2x\,f(x) = 2x\,[f(x)]^2$$
$$2x\,(f(x) - [f(x)]^2) = 0$$

which means that $x = 0$ or $f(x) - [f(x)]^2 = 0$. In the case when $x = 0$, $f(x)$ can be any real value; it only has to exist. When $x \neq 0$,

$$f(x) - [f(x)]^2 = 0$$
$$[1 - f(x)]\,f(x) = 0$$
$$f(x) = 0 \text{ or } 1$$

it allows $f(x)$ to change values between 0 and 1 arbitrarily for different x. However, consider $x, y \neq 0$ such that $f(x) = 0$ and $f(y) = 1$. In this case, $x\,f(y) + y\,f(x) = (x+y)\,f(x)\,f(y)$ reduces to $x = 0$, which is a contradiction. Therefore, in the domain $x \neq 0$, $f(x)$ must be a constant, $f(x) \equiv 1$ or $f(x) \equiv 0$ for $x \neq 0$.

when $f(x) = 0$ for $x = 0$, then use $x = 1$, $y = 0$ and obtain $f(0) = 0$.
when $f(x) = 1$ for $x \neq 0$, then $f(0) = a$, arbitrarily, satisfies the equation.
Thus $f(x) = \begin{cases} a, & \text{for } x = 0 \\ 1, & \text{for } x \neq 0 \end{cases}$ is a discontinuous solution for $a \neq 1$, and $f(x) \equiv 0$, $f(x) \equiv 1$ are the only continuous solution.

It is easy to see that those functions are indeed solutions.

Also, at least partially solved by:

<u>Undergraduates</u>: Mehmet Demirci (Jr. CS), Tu Tam Nguyen Phan (MATH.)

<u>Graduates</u>: Sridhar Maddipati (Ch.E.), Ashish Rao (ECE), Qi Xu (Ch.E.)

<u>Others</u>: P. Chebulu (CMU, Pittsburg), Georges Ghosn (Quebec), Steven Landy (IUPUI)