PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2004 Series)

Problem: Find all positive integers m, n such that

$$2^m = 3^n + 5. \qquad (1)$$

Solution (by the Panel)

It is easy to see that $m = 3$, $n = 1$; $m = 5$, $n = 3$ are all solutions with $m \leq 5$. We will show that they are the only ones.

Let $m > 5$. Then $3^n + 5$ is divisible by $2^6 = 64$. On the other hand, $3^{16} \equiv 1 \mod 64$, and a direct calculation shows that $3^{11} \equiv -5$, and $3^k \not\equiv -5$ for all other $k = 0, \ldots, 15$. So, n must be of the form:

$$n = 16k + 11.$$

Now, if we divide (1) by 17, using $3^{16} \equiv 1 \mod 17$, we get $3^{16k+11} \equiv 3^{11} \equiv 7 \mod 17$.

Therefore, $2^m \equiv 12 \mod 17$.

On the other hand, the possible remainders of 2^m, divided by 17 are

$$2,\ 4,\ 8,\ 16,\ 15,\ 13,\ 9,\ 1$$

and in particular, $2^8 \equiv 1 \mod 17$.

Therefore $2^m \equiv 12 \mod 17$ is impossible.

There were no acceptable solutions presented.