PROBLEM OF THE WEEK

Solution of Problem No. 6 (Fall 2004 Series)

Problem: Determine the linear function $L(x) = ax + b$ for which

$$\max_{0\le x\le 1} |x\ln x - L(x)|$$

is least. Prove your answer. Here, $x \ln x$ is extended to $x = 0$ by continuity.

Solution (by the Panel)

Denote $f(x) = x\ln x$, $\varepsilon(L) = \max_{0\le x\le 1} |f(x) - L(x)|$, where $L(x)$ is a linear function. We will show that the answer is $L_0(x) = -1/(2e)$ (then $\varepsilon(L_0) = 1/(2e)$).

Let $L(x)$ be any linear function with $\varepsilon(L) < 1/(2e)$. Then $|L(0)| < 1/(2e)$, $|L(1)| < 1/(2e)$ because $f(0) = f(1) = 0$. Since $L(x)$ is linear, then $|L(1/e)| < 1/(2e)$ as well. Since $f(1/e) = -1/e$, we get $\varepsilon(L) > 1/(2e)$, which is a contradiction. On the other hand, if $\varepsilon(L) = 1/(2e)$, then the same arguments yield $L = L_0$.

Therefore, $L_0(x)$ is the unique linear function minimizing $\varepsilon(L)$.

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