PROBLEM OF THE WEEK
Solution of Problem No. 7 (Fall 2004 Series)

Problem: Prove (without calculus) that $\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}\geq 3\sqrt{3}$ if α,β,γ are the angles of a triangle.

Solution (by the Panel)

Let p be the half-perimeter of the triangle, and let a,b,c be its sides. Denote by r the radius of the inscribed circle. Then

$$\cot\alpha/2=\frac{a}{2r},\qquad \cot\beta/2=\frac{b}{2r},\qquad \cot\gamma/2=\frac{c}{2r}.$$

Therefore, $M := \cot\alpha/2+\cot\beta/2+\cot\gamma/2 = p/r = \dfrac{p^2}{A}$, where A is the area of the triangle. We use the fact that for a triangle with a fixed perimeter, the area is maximized when $a=b=c$, and then $A=\dfrac{\sqrt{3}}{4}a^2=\dfrac{\sqrt{3}}{9}p^2$, therefore $M\geq 9/\sqrt{3}=3\sqrt{3}$.

Now, we will prove the statement above. It is known that

$$A=\sqrt{p(p-a)(p-b)(p-c)}.$$

Therefore, $A\leq\sqrt{p}\left(\dfrac{(p-a)+(p-b)+(p-c)}{3}\right)^{3/2}=\dfrac{p^2}{3\sqrt{3}}=\dfrac{\sqrt{3}}{9}p^2$, and this is the inequality we used (it turns into an equality if and only if $p-a=p-b=p-c\Longleftrightarrow a=b=c$).

<u>Second Solution</u> (provided by T. Pollom, HS student, edited by the panel)

The function $\cot x$ is convex on $(0,\pi/2)$ because its second derivative is positive. Therefore,

$$\frac{1}{3}\left(\cot\alpha/2+\cot\beta/2+\cot\gamma/2\right)\geq\cot\left(\frac{\alpha/2+\beta/2+\gamma/2}{3}\right)=\cot\frac{\pi}{6}=\sqrt{3}.$$

Solved by:

<u>Undergraduates</u>: Yuandong Tian (Sr. ECE)

<u>Graduates</u>: George Hassapis (MATH), Ashish Rao (ECE)

<u>Others</u>: P. Chebulu (CMU, Pittsburg), Prithwijit De, Georges Ghosn (Quebec), Steven Landy (IUPUI), Aaditya Muthukumaran (Chennai, India), Thomas Pollom (HS student, Indianapolis), S. Sanjiv (ECE, Waterloo)