PROBLEM OF THE WEEK

Solution of Problem No. 8 (Fall 2004 Series)

Problem: Are there sets S of more than four points in three-dimensional space such that any four points of S are the vertices of a tetrahedron of volume 1?

Solution (by the Panel)

We will prove that the answer is "No". Assume that we have 5 points with that property: A_1, A_2, A_3, A_4, A_5. We will show first that without loss of generality, we can assume that $A_1 = (0,0,0)$, $A_2 = (1,0,0)$, $A_3 = (0,1,0)$, $A_4 = (0,0,1)$. Indeed, any linear transformation $y = Bx + c$, where B is a 3×3 matrix, c is a fixed vector, x are the old coordinates, y are the new ones; changes the volume of each body by the fixed factor $|\det B|$. Therefore, we can choose such transformation that would change the coordinates of A_1, A_2, A_3, A_4 as shown above, and then the volumes of all tetrahedrons with vertices among those 5 points will be equal (to $1/6$).

Let $A = (x, y, z)$. Since Vol $(A_1A_2A_3A_5) = 1$ we have $z = 1$ or $z = -1$. In the same way we obtain $y = \pm 1$, $z = \pm 1$. Next, Vol $(A_2A_3A_4A_5) = 1$ implies that A_5 lies on a plane parallel to the one through A_2, A_3, A_4 and passing through A_1, or symmetric to the latter about $A_2A_3A_4$. In other words,

$$x + y + z = 0 \quad \text{or} \quad x + y + z = 2 \qquad (1)$$

Since we know that $(x, y, z) = (\pm 1, \pm 1, \pm 1)$, we see that there is no combination of the signs above that would satisfy (1).

Solved by:

Undergraduates: None

Graduates: None

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