PROBLEM OF THE WEEK
Solution of Problem No. 9 (Fall 2004 Series)

Problem: Show that for any positive integer n, the number $(1+\sqrt{2})^n$ differs from an integer by less than $1/2^n$.

Solution (by Georges Ghosn, Quebec, edited by the Panel)

Using the binomial theorem:

$$(1+\sqrt{2})^n + (1-\sqrt{2})^n = \sum_{p=0}^{n} \binom{n}{p} (\sqrt{2})^p + \sum_{p=0}^{n} \binom{n}{p} (-\sqrt{2})^p = 2 \sum_{p=0}^{2p \leq n} \binom{n}{2p} 2^p$$

is an integer. Since $|(1-\sqrt{2})^n| = \dfrac{1}{(1+\sqrt{2})^n} < \dfrac{1}{2^n}$, we deduce that $(1+\sqrt{2})^n$ differs from an integer by less than $\dfrac{1}{2^n}$.

Also solved by:

Undergraduates: Al-Sharif Al-Housseiny (So. CE), Yuandong Tian (Sr. ECE)

Graduates: Ashish Rao (ECE), Amit Shirsat (CS)

Others: P. Chebulu (CMU, Pittsburg), Byungsoo Kim (Seoul Natl. Univ.), Steven Landy (IUPUI), Graeme McRae, Naming Mammadov (Azerbaijan), Thomas Pollom (HS student, Indianapolis), Jim Schofield (Rosemont HS, Barnsville, MN)

Update on Problem 8: This problem was solved at least partially, also by Byungsoo Kim and Thomas Pollom. The panel appologizes for not listing their names under the solution of Problem 8.