PROBLEM OF THE WEEK

Solution of Problem No. 11 (Fall 2004 Series)

Problem: Show that there exists a constant C, such that, for any polynomal P of degree 2004, we have

$$|P(1) - P'(1) + P(-1) + P'(-1)| \leq C \int_{-1}^{1} |P(x)|\, dx,$$

where $P' = dP/dx$.

For extra credit, show that $C \geq 4,000,000$.

Solution

Let

$$P = a_0 + a_1 x + \cdots + a_{2004} x^{2004},$$

and consider the function

$$f(a_0, a_1, \ldots, a_{2004}) = \frac{|P(1) - P'(1) + P(-1) + P'(-1)|}{\int_{-1}^{1} |P(x)|\, dx}$$

In other words, we regard the right-hand side as a function of the coefficients of P. The function f is homogeneous of order zero, i. e.,

$$f(ta_0, ta_1, \ldots, ta_{2004}) = f(a_0, a_1, \ldots, a_{2004}), \qquad \forall t \neq 0$$

Next, f is continuous on the unit sphere

$${a_0}^2 + {a_1}^2 + \cdots + {a_{2004}}^2 = 1,$$

therefore it has a maximal value there, let us call it C. By the homogeneity, we also have $f \leq C$ for any other $(a_0, a_1, \ldots, a_{2004}) \neq 0$. This completes the proof. Note that in the proof we used the fact that $\int_{-1}^{1} |P(x)| dx = 0 \iff P \equiv 0$.

To show that any such constant is greater or equal to 4,000,000, apply the inequality with $P(x) = x^{2004}$.

Solved by:

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