PROBLEM OF THE WEEK

Solution of Problem No. 13 (Fall 2004 Series)

Problem: For $k \geq 2$ and $b \geq 2\tan\dfrac{\pi}{2k}$, prove that, up to congruence, there is a unique polygon with $2k$ sides, each of length b, circumscribed (once) about the unit circle.

The formulation of the problem is wrong. The correct one is the following:

Let $k \geq 2$. Prove that, up to congruence, there is a unique polygon with $2k$ sides, each of length b, circumscribed (once) about the unit circle, if

$$2\tan\frac{\pi}{2k} \leq b < \tan\frac{\pi}{k}. \tag{1}$$

If $k = 2$, then the second inequality above is reduced to $b < \infty$, i. e., the only requirement then is $2 \leq b$.

Solution (by the Panel)

Let A_1, A_2, A_3 be three consecutive vertices of the polygon, if it exists. Let M_1 and M_2 be the common points of A_1A_2 and A_2A_3 with the circle, respectively. Then it is easy to show that $\angle M_1OA_2 = \angle M_2OA_2$ (let us call it α), and $\angle A_1OM_1 = \angle A_3OM_2$ (let us call it β), where O is the center of the circle. We can repeat those arguments for A_2, A_3 and A_4, etc. As a consequence of that, we get that $\angle A_jOA_{j+1} = \alpha + \beta$, $j = 1, \dots, 2k$ with the convention $A_{2k+1} = A_1$. Thus, $2k(\alpha + \beta) = 2\pi$, so

$$\alpha + \beta = \frac{\pi}{k}, \qquad \alpha > 0, \quad \beta > 0. \tag{2}$$

We also have

$$\tan\alpha + \tan\beta = b. \tag{3}$$

On the other hand, it is easy to see that if we have a solution of (2), (3), then there exists a polygon with the required properties. Each solution (α_0, β_0) corresponds to polygons related to each other by rotation; on the other hand (β_0, α_0) is also a solution, and it corresponds to polygons obtained from the first group by symmetry about a line passing through O.

So the problem reduces to the following: Prove that under the condition (1), there is unique solution of (2), (3), up to the symmetry $(\alpha, \beta) \mapsto (\beta, \alpha)$. The latter follows from analysis of the function

$$f(\alpha) = \tan\alpha + \tan(\pi/k - \alpha), \quad 0 \leq \alpha \leq \pi/k.$$

The function f is positive, attains a minimum $f_{\min} = 2\tan\frac{\pi}{2k}$ at $\alpha = \frac{\pi}{2k}$, it is decreasing for $0 < \alpha < \frac{\pi}{2k}$ and increasing for $\frac{\pi}{2k} < \alpha < \frac{\pi}{k}$. At the endpoints, $f(0) = f(\frac{\pi}{k}) = \tan\frac{\pi}{k}$ (which equals $+\infty$, if $k = 2$), so $f_{max} = \tan\frac{\pi}{k}$. Now, $f(\alpha) = b$ is solvable for any $b \in [f_{min}, f_{max}]$, and the requirement that $\alpha > 0, \beta > 0$, actually implies that we must have $b < f_{max}$. Under the condition $b \in [f_{\min}, f_{max}]$, which is equivalent to (1), there are two symmetric roots in $(0, \pi/k)$, that coincide if $b = 2\tan\frac{\pi}{2k}$, and this is what we had to prove.

Remark. If $k > 2$, and $b = \tan\frac{\pi}{k}$, such a (degenerate) polygon still exists. It is a regular polygon with k sides but if we count the points of contact with the circle as vertices, it would have $2k$ sides. Example: a square circumscribed about the unit circle, with the points of contact considered as additional vertices.

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