## PROBLEM OF THE WEEK Solution of Problem No. 1 (Fall 2005 Series)

**Problem:** Evaluate the product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

Solution (by the Panel)

Let  $P_N = \prod_{n=2}^N \frac{n^3 - 1}{n^3 + 1}$ . We need to show that  $P_N$  converges and to find lim  $P_N$ . Since  $\frac{n^3 - 1}{n^3 + 1} = \frac{n - 1}{n + 1} \cdot \frac{n^2 + n + 1}{n^2 - n + 1}$ , we have

(1) 
$$P_{N} = \prod_{n=2}^{N} \frac{n-1}{n+1} \prod_{n=2}^{N} \frac{n^{2}+n+1}{n^{2}-n+1} = \left(\frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdot \dots \cdot \frac{N-1}{N+1}\right) \left(\frac{7}{3} \cdot \frac{13}{7} \cdot \frac{21}{13} \cdot \dots \cdot \frac{N^{2}+N+1}{N^{2}-N+1}\right)$$
(3) 
$$= \frac{2}{N(N+1)} \cdot \frac{N^{2}+N+1}{3}.$$

The reason for this equality is many cancelations but just because the first 3-4 terms cancel beautifully, does not mean that all of them will. Below, we prove that (1) and (3) are indeed equal.

 $\operatorname{Set}$ 

$$P'_{N} = \prod_{n=2}^{N} \frac{n-1}{n+1}, \quad P''_{N} = \prod_{n=2}^{N} \frac{n^{2}+n-1}{n^{2}-n+1}$$

We need to prove that

(4) 
$$P'_N = \frac{2}{N(N+1)}, \quad P''_N = \frac{N^2 + N + 1}{3}, \quad N = 2, 3, \dots$$

We will use mathematical induction and will show how to prove the second equality only.

First,  $P_2'' = \frac{7}{3}$ , so it is true for N = 2.

Assume if it is true for N = k, i.e.

$$P_k^{''} = \frac{k^2 + k + 1}{3}.$$

We will show that it is true for N = k + 1. Indeed,

$$P_{k+1}^{''} = \frac{(k+1)^2 + (k+1) + 1}{(k+1)^2 - (k+1) + 1} P_k = \frac{(k+1)^2 + (k+1) + 1}{k^2 + k + 1} \cdot \frac{k^2 + k + 1}{3}$$
$$= \frac{(k+1)^2 + (k+1) + 1}{3}.$$

Therefore, the second statement in (4) holds for N = k + 1 as well.

By the principle of math induction, it is true for any N.

Now, take the limit  $N \to \infty$  in (3) to get

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1} = \frac{2}{3}$$

Solved by:

<u>Undergraduates</u>: Brian Bright (Sr. Stat), Chris Gianopoulous (Fr.), Akira Matsudaira (Sr. ECE), Arman Sabbaghi (Fr. Math & Stat), Justin Woo (Sr. ECE)

<u>Graduates</u>: Shreekant Gayaka (ME), Miguel Hurtado (ECE), Jose Lugo (MA), Amit Shirsat (CS)

<u>Others</u>: George Barnett (Woodland CC, CA), Brett Coryell (Purdue Staff), Prithwijit De (Ireland), Elefteri & Sarat (Johns Hopkins), Georges Ghosn (Quebec), Steven Landy (IUPUI Physics staff), Aaditya Muthukumaran (Chennai, India), Sridharakusmar Narasimhan (Postsdam, NY), David Padgett (NCSU, Raleigh, NC), A. Plaza (ULPGC, Spain), Rob Pratt (Raleigh, NC), Ashish Rao, David Stigant (Teacher, Houston, TX), Sunder Thiroupapuliyur (Santa Clara, CA), Daniel Vacaru (Pitesti, Romania), Jim Vaught (Lafayette, IN)