

PROBLEM OF THE WEEK
Solution of Problem No. 10 (Fall 2005 Series)

Problem: Which positive integers n are expressible in at least one way as the sum of two or more consecutive positive integers? Prove your answer.

Solution (by Prithwijit De & Sonali Dasgupta, U.C.C, Republic of Ireland)

Claim: All positive integers n except the powers of 2 can be written as sum of two or more consecutive positive integers. Suppose n can be written as a sum of ℓ consecutive numbers beginning with $(k + 1)$. Then

$$n = (k + 1) + (k + 2) + \dots + (k + \ell) = \frac{\ell(2k + \ell + 1)}{2}.$$

Now, one of ℓ or $(2k + \ell + 1)$ is odd (and the other one is even). Therefore, n is not a power of 2.

Conversely, let n be a positive integer with an odd factor. Since n has an odd factor, so does $2n$, and we can write $2n = f_1 f_2$ where one of f_1 or f_2 is odd, the other one is even, and $1 < f_1 < f_2$. Let $k = \frac{f_2 - f_1 - 1}{2}$, $\ell = f_1$, then $f_2 = 2k + \ell + 1$, so that $n = \frac{f_1 f_2}{2} = \frac{\ell(2k + \ell + 1)}{2} = (k + 1) + (k + 2) + (k + 3) + \dots + (k + \ell)$.

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