PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2005 Series)

Problem: Let $P(x)$ be a polynomial of odd degree with real coefficients. Let a be a fixed real number and assume that $P''(a) \neq 0$. Prove that for any $t \in \left(0, \frac{1}{2}\right)$ there exists $b \neq a$ such that

$$\frac{P(b) - P(a)}{b - a} = P'\Big(tb + (1-t)a\Big).$$

Solution (by Bob Hanek)

Consider the polynomial $Q(x) = P(x+a) - P(a) - xP'(tx+a)$. Since P is of odd degree and $P''(a) \neq 0$, P is of at least degree three and

$$Q'(x) = P'(x+a) - P'(tx+a) - txP''(tx+a) \qquad \text{and}$$
$$Q''(x) = P''(x+a) - 2tP''(tx+a) - t^2xP'''(tx+a).$$

From which it follows that $Q(0) = Q'(0) = 0$ and $Q''(0) = (1-2t)P''(a) \neq 0$. Consequently, $Q(x) = x^2R(x)$ for some odd degree polynomial, $R(x)$, with $R(0) \neq 0$. Since $R(x)$ is of odd degree, it must have at least one real zero, and since $R(0) \neq 0$, this implies that there exists a real number $\xi \neq 0$ such that $R(\xi) = 0$. It follows that $Q(\xi) = 0$ and therefore

$$\frac{P(\xi + a) - P(a)}{\xi} = P'(t\xi + a).$$

The result follows by taking $b = \xi + a$.

At least partially solved by:

Stephen Casey (Ireland), Prithwijit De (Ireland), Georges Ghosn (Quebec), Sridharakusmar Narasimhan (Postsdam, NY), Steve Spindler (Chicago), David Stigant (Teacher, Houston, TX)